

VARIATIONS OF MIXED HODGE STRUCTURE AND SEMI-POSITIVITY THEOREMS

OSAMU FUJINO AND TARO FUJISAWA

ABSTRACT. We discuss the variations of mixed Hodge structures arising from the mixed Hodge structures on compact support cohomology groups of simple normal crossing pairs. We show that they are graded polarizable admissible variations of mixed Hodge structures. Then we prove a generalization of the Fujita–Kawamata semi-positivity theorem.

CONTENTS

| | |
|---|----|
| 1. Introduction | 1 |
| 2. Preliminaries | 8 |
| 3. Generalities on variation of mixed Hodge structure | 12 |
| 4. Variation of mixed Hodge structure of geometric origin | 35 |
| 5. Higher direct images of log canonical divisors | 53 |
| 6. Semi-positivity theorem | 62 |
| 7. Vanishing and torsion-free theorems | 72 |
| 8. Examples | 76 |
| References | 82 |

1. INTRODUCTION

Let X be a simple normal crossing divisor on a smooth variety M and let B be a simple normal crossing divisor on M such that $X + B$ is simple normal crossing on M and that X and B have no common irreducible components. Then the pair (X, D) , where $D = B|_X$, is a typical example of simple normal crossing pairs. In this situation, a

Date: 2012/3/30, version 3.00.

2010 *Mathematics Subject Classification.* Primary 14D07; Secondary 14C30, 14E30, 32G20.

Key words and phrases. variation of mixed Hodge structure, compact support cohomology groups, canonical extensions of Hodge bundles, semi-positivity theorems, vanishing theorems, torsion-free theorem, simple normal crossing varieties, weak positivity, canonical bundle formula.

stratum of (X, D) is an irreducible component of $T_{i_1} \cap \cdots \cap T_{i_k} \subset X$ for some $\{i_1, \dots, i_k\} \subset I$, where $X + B = \sum_{i \in I} T_i$ is the irreducible decomposition of $X + B$. For the precise definition of simple normal crossing pairs, see Definition 2.6 below. We note that simple normal crossing pairs very often appear in the study of the log minimal model program for higher dimensional algebraic varieties with bad singularities. The first author has already investigated the mixed Hodge structures on compact support cohomology groups $H_c^\bullet(X \setminus D, \mathbb{Q})$ in [F7, Chapter 2] to obtain various vanishing theorems. Roughly speaking, this paper is a continuation of [F7, Chapter 2] and is devoted to the study of the variations of the above mentioned mixed Hodge structures. We show that they are graded polarizable admissible variations of mixed Hodge structures. Then we prove a generalization of the Fujita–Kawamata semi-positivity theorem. Our formulation of the Fujita–Kawamata semi-positivity theorem is slightly different from Kawamata’s original one. However, it is more suited for our studies of simple normal crossing pairs.

The following theorem is a corollary of Theorem 5.1 and Theorem 5.3, which are our main results of this paper (cf. [Kw1, Theorem 5], [Ko2, Theorem 2.6], [N1, Theorem 1], [F4, Theorems 3.4 and 3.9], [Kw3, Theorem 1.1], and so on). It is an answer to the question raised by Valery Alexeev and Christopher Hacon.

Theorem 1.1 (Semi-positivity theorem (cf. Theorem 5.1 and Theorem 5.3)). *Let (X, D) be a simple normal crossing pair such that D is reduced and let $f : X \rightarrow Y$ be a projective surjective morphism onto a smooth complete algebraic variety Y . Assume that every stratum of (X, D) is dominant onto Y . Let Σ be a simple normal crossing divisor on Y such that every stratum of (X, D) is smooth over $Y_0 = Y \setminus \Sigma$. Then $R^p f_* \omega_{X/Y}(D)$ is locally free for every p . We put $X_0 = f^{-1}(Y_0)$, $D_0 = D|_{X_0}$, and $d = \dim X - \dim Y$. We further assume that all the local monodromies on $R^{d-i}(f|_{X_0 \setminus D_0})! \mathbb{Q}_{X_0 \setminus D_0}$ around Σ are unipotent. Then we obtain that $R^i f_* \omega_{X/Y}(D)$ is a semi-positive (in the sense of Fujita–Kawamata) locally free sheaf on Y .*

We note the following definition.

Definition 1.2 (Semi-positivity in the sense of Fujita–Kawamata). A locally free sheaf \mathcal{E} on a complete algebraic variety X is said to be semi-positive (in the sense of Fujita–Kawamata) if and only if $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ is nef on $\mathbb{P}_X(\mathcal{E})$.

The semi-positivity of $R^i f_* \omega_{X/Y}(D)$ in Theorem 1.1 follows from a purely Hodge theoretic semi-positivity theorem: Theorem 6.21. In

the proof of Theorem 1.1, we use the semi-positivity of $(\mathrm{Gr}_F^a \mathcal{V})^*$ in Theorem 1.3. We do not need the semi-positivity of $F^b \mathcal{V}$ in Theorem 1.3 for Theorem 1.1. For more details, see the discussion in 1.7 below.

Theorem 1.3 (Hodge theoretic semi-positivity theorem (cf. Theorem 6.21)). *Let X be a smooth complete complex algebraic variety, D a simple normal crossing divisor on X , \mathcal{V} a locally free \mathcal{O}_X -module of finite rank equipped with a finite increasing filtration W and a finite decreasing filtration F . We assume the following:*

- (1) $F^a \mathcal{V} = \mathcal{V}$ and $F^{b+1} \mathcal{V} = 0$ for some $a < b$.
- (2) $\mathrm{Gr}_F^p \mathrm{Gr}_m^W \mathcal{V}$ is a locally free \mathcal{O}_X -module of finite rank for all m, p .
- (3) For all m , $\mathrm{Gr}_m^W \mathcal{V}$ admits an integrable logarithmic connection ∇_m with the nilpotent residue morphisms which satisfies the conditions $\nabla_m(F^p \mathrm{Gr}_m^W \mathcal{V}) \subset F^{p-1} \mathrm{Gr}_m^W \mathcal{V}$ for all p .
- (4) The pair $(\mathrm{Gr}_m^W \mathcal{V}, F, \nabla_m)|_{X \setminus D}$ underlies a polarizable variation of \mathbb{R} -Hodge structure of weight m for every integer m .

Then $(\mathrm{Gr}_F^a \mathcal{V})^*$ and $F^b \mathcal{V}$ are semi-positive.

In this paper, we concentrate on the Hodge theoretic aspect of the Fujita–Kawamata semi-positivity theorem (cf. [Kw1], [Ko2], [N1], and [F4]). On the other hand, there are many results related to the Fujita–Kawamata semi-positivity theorem from the analytic viewpoint (cf. [Ft], [Be], [BeP], [MT], and so on). Note that Griffiths’ pioneering work on the variation of Hodge structure (cf. [Gf]) is a starting point of the Fujita–Kawamata semi-positivity theorem.

As a special case of Theorem 1.1, we obtain the following theorem: Theorem 5.8. It means that the formulation of Theorem 1.1 is a reasonable generalization of the Fujita–Kawamata semi-positivity theorem.

Theorem 1.4 (cf. [Kw1, Theorem 5], [Kw2, Theorem 2], [Ko2, Theorem 2.6], [N1, Theorem 1]). *Let $f : X \rightarrow Y$ be a projective morphism between smooth complete algebraic varieties which satisfies the following conditions:*

- (i) *There is a Zariski open subset Y_0 of Y such that $\Sigma = Y \setminus Y_0$ is a simple normal crossing divisor on Y .*
- (ii) *We put $X_0 = f^{-1}(Y_0)$ and $f_0 = f|_{X_0}$. Then f_0 is smooth.*
- (iii) *The local monodromies of $R^{d+i}(f_0)_* \mathbb{C}_{X_0}$ around Σ are unipotent, where $d = \dim X - \dim Y$.*

Then $R^i f_* \omega_{X/Y}$ is a semi-positive locally free sheaf on Y .

We note that Theorem 1.4 was first proved by Kawamata (cf. [Kw1, Theorem 5]) under the extra assumptions that $i = 0$ and that f has

connected fibers. The above statement of Theorem 1.4 is a combination of [Kw2, Theorem 2] with [Ko2, Theorem 2.6] or [N1, Theorem 1]. We also note that, by the Poincaré–Verdier duality, $R^{d+i}(f_0)_*\mathbb{C}_{X_0}$ is the dual local system of $R^{d-i}(f_0)_*\mathbb{C}_{X_0}$ in Theorem 1.4. In [Kw1], [Ko2], and [N1], the variation of Hodge structure on $R^{d+i}(f_0)_*\mathbb{C}_{X_0}$ is investigated for the proof of Theorem 1.4. On the other hand, in this paper, we concentrate on the variation of Hodge structure on $R^{d-i}(f_0)_*\mathbb{C}_{X_0}$ for Theorem 1.4.

In [Kw3], Kawamata obtained a result similar to Theorem 1.1 (see [Kw3, Theorem 1.1]). It is not surprising because both [Kw3] and this paper grew from a question raised by Valery Alexeev and Christopher Hacon. Kawamata’s proof in [Kw3] depends on the theory of variation of mixed Hodge structure. Kawamata pointed out that the underlying local system of the variation of mixed Hodge structure discussed in [Kw3] does not come from a locally trivial family of topological spaces. We do not know whether the variation of mixed Hodge structure in [Kw3] has good properties or not. Our proof of Theorem 1.1 also depends on the theory of variation of mixed Hodge structure. However, our variation of mixed Hodge structure is different from Kawamata’s. We show that it is graded polarizable and admissible. Precisely speaking, we prove the following theorem.

Theorem 1.5 (GPVMHS arising from mixed Hodge structures on compact support cohomology groups (cf. Theorem 4.20)). *Let X be a complex variety, $Z \subset X$ a closed subvariety of X and $f : X \rightarrow Y$ a projective surjective morphism to a smooth complex variety Y . Then there exists a Zariski open dense subset Y_0 of Y such that $R^n(f|_{X \setminus Z})_!\mathbb{Q}_{X \setminus Z}|_{Y_0}$ underlies an admissible graded polarizable variation of \mathbb{Q} -mixed Hodge structure for every n .*

Although it seems to follow from the theory of mixed Hodge modules by Morihiko Saito [Sa2], we give a detailed proof in Section 4, because we need an explicit description of the Hodge filtration in Theorem 1.1. For related results, see [St1], [St2], [SZ], [Ks], [dB], [E1], [NA], [GN], and so on.

The following example shows that the assumption (2) in Theorem 1.3 is indispensable. In the proof of Theorem 1.1, the admissibility of the graded polarizable variation of \mathbb{Q} -mixed Hodge structure on $R^{d-i}(f|_{X_0 \setminus D_0})_!\mathbb{Q}_{X_0 \setminus D_0}$, which is proved in Theorem 1.5, assures us the existence of the extension of the Hodge filtration satisfying the assumption (2). Although the existence of such extension seems to be nontrivial for Kawamata’s variation of mixed Hodge structure, there

are no discussions in [Kw3]. For related examples, see [SZ, (3.15) and (3.16)].

Example 1.6. Let \mathbb{V} be a 2-dimensional \mathbb{Q} -vector space with basis $\{e_1, e_2\}$. We give an increasing filtration W on \mathbb{V} by $W_{-1}\mathbb{V} = 0$, $W_0\mathbb{V} = W_1\mathbb{V} = \mathbb{Q}e_1$, and $W_2\mathbb{V} = \mathbb{V}$. The constant sheaf on \mathbb{P}^1 whose fibers are \mathbb{V} is denoted by the same letter \mathbb{V} , on which an increasing filtration W is given as above. We consider $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1} \otimes \mathbb{V} = \mathcal{O}_{e_1} \oplus \mathcal{O}_{e_2}$ on \mathbb{P}^1 . We set a decreasing filtration F on $\mathcal{V}|_{\mathbb{C}^*}$ by

$$F^0(\mathcal{V}|_{\mathbb{C}^*}) = \mathcal{V}|_{\mathbb{C}^*}, \quad F^1(\mathcal{V}|_{\mathbb{C}^*}) = \mathcal{O}_{\mathbb{C}^*}(t^{-1}e_1 + e_2), \quad F^2(\mathcal{V}|_{\mathbb{C}^*}) = 0,$$

where t is the coordinate function of $\mathbb{C} \subset \mathbb{P}^1$. We can easily check that $((\mathbb{V}, W)|_{\mathbb{C}^*}, (\mathcal{V}|_{\mathbb{C}^*}, F))$ is a graded polarizable variation of \mathbb{Q} -mixed Hodge structure on \mathbb{C}^* . In this case, we can not extend the Hodge filtration F on $\mathcal{V}|_{\mathbb{C}^*}$ to the filtration F on \mathcal{V} satisfying the assumption (2) in Theorem 1.3. In particular, the above variation of \mathbb{Q} -mixed Hodge structure is not admissible.

We note that we can extend the Hodge filtration F on $\mathcal{V}|_{\mathbb{C}^*}$ to the filtration F on \mathcal{V} such that $F^2\mathcal{V} = 0$, $F^1\mathcal{V} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$, and $F^0\mathcal{V} = \mathcal{V}$ with $\mathrm{Gr}_F^0\mathcal{V} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$. In this case, $F^1\mathcal{V}$ and $(\mathrm{Gr}_F^0\mathcal{V})^*$ are not semi-positive. This means that a naive generalization of the Fujita–Kawamata semi-positivity theorem for graded polarizable variations of \mathbb{Q} -mixed Hodge structure is false.

Our treatment of Theorem 1.1 is completely different from Kawamata’s in [Kw3] and is very natural from the viewpoint developed by the first author in [F7, Chapter 2].

We give a sketch of the proof of Theorem 1.1 for the reader’s convenience.

1.7 (Sketch of the proof of Theorem 1.1). In Theorem 1.1, we see that the local system $R^{d-i}(f|_{X_0 \setminus D_0})! \mathbb{Q}_{X_0 \setminus D_0}$ underlies an admissible variation of \mathbb{Q} -mixed Hodge structure by Theorem 1.5 and its proof (see Section 4). Let \mathcal{V} be the canonical extension of $(R^{d-i}(f|_{X_0 \setminus D_0})! \mathbb{Q}_{X_0 \setminus D_0}) \otimes \mathcal{O}_{Y_0}$. Then we can prove $R^{d-i}f_* \mathcal{O}_X(-D) \simeq \mathrm{Gr}_F^0 \mathcal{V}$ where F is the canonical extension of the Hodge filtration. Note that the *admissibility* ensures the existence of the canonical extensions of the Hodge bundles (cf. Proposition 3.13). We also note that we use an explicit description of the canonical extension of the Hodge filtration in order to prove $R^{d-i}f_* \mathcal{O}_X(-D) \simeq \mathrm{Gr}_F^0 \mathcal{V}$ when Y is a curve. By the Grothendieck duality, we obtain $R^i f_* \omega_{X/Y}(D) \simeq (\mathrm{Gr}_F^0 \mathcal{V})^*$. Therefore, $R^i f_* \omega_{X/Y}(D)$ is semi-positive by Theorem 1.3. It is very important to note that the local system $R^{d-i}(f|_{X_0 \setminus D_0})! \mathbb{Q}_{X_0 \setminus D_0}$ is not necessarily the dual local system

of $R^{d+i}(f|_{X_0 \setminus D_0})_* \mathbb{Q}_{X_0 \setminus D_0}$ because X is not a smooth variety but a simple normal crossing variety. Thus it seems to be indispensable to investigate the variation of mixed Hodge structure on $R^{d-i}(f|_{X_0 \setminus D_0})! \mathbb{Q}_{X_0 \setminus D_0}$ for the semi-positivity of $R^i f_* \omega_{X/Y}(D)$. See also the discussion in 1.8 below.

One of the main differences between our main theorems (cf. Theorem 5.1 and Theorem 5.3) and other known results is to use mixed Hodge structures on compact support cohomology groups.

We quickly explain the reason why we use mixed Hodge structures on compact support cohomology groups.

1.8 (Mixed Hodge structures on compact support cohomology groups). Let X be a smooth projective variety and let D be a simple normal crossing divisor on X . After Iitaka introduced the notion of logarithmic Kodaira dimension, $\mathcal{O}_X(K_X + D)$ plays important roles in the birational geometry, where K_X is the canonical divisor of X . In the traditional birational geometry, $\mathcal{O}_X(K_X + D)$ is recognized to be $\Omega_X^{\dim X}(\log D)$. Therefore, the Hodge to de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega_X^p(\log D)) \Rightarrow H^{p+q}(X \setminus D, \mathbb{C})$$

arising from the mixed Hodge structures on $H^\bullet(X \setminus D, \mathbb{C})$ is useful. The first author recognizes $\mathcal{O}_X(K_X + D)$ as

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-D), \mathcal{O}_X(K_X))$$

or

$$\mathcal{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-D), \omega_X^\bullet)[- \dim X]$$

where $\omega_X^\bullet = \mathcal{O}_X(K_X)[\dim X]$ is the dualizing complex of X . Furthermore, $\mathcal{O}_X(-D)$ can be interpreted as the 0-th term of the complex

$$\Omega_X^\bullet(\log D) \otimes \mathcal{O}_X(-D).$$

By this observation, we can use the following Hodge to de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega_X^p(\log D) \otimes \mathcal{O}_X(-D)) \Rightarrow H_c^{p+q}(X \setminus D, \mathbb{C})$$

arising from the mixed Hodge structures on compact support cohomology groups $H_c^\bullet(X \setminus D, \mathbb{C})$ and obtain various powerful vanishing theorems. For the details and many applications, see [F8], [F10], [F11, Section 5], [F12], [F15], and [F7, Chapter 2]. Therefore, it is very natural to consider the variations of such mixed Hodge structures.

In this paper, we consider the variations of mixed Hodge structures arising from the mixed Hodge structures on compact support cohomology groups of families of quasi-projective simple normal crossing pairs. We will show that they are graded polarizable and admissible.

We summarize the contents of this paper. Section 2 is a preliminary section. We discuss divisors on simple normal crossing varieties in detail. This section is indispensable to discuss reducible varieties and divisors on them in subsequent sections. Section 3 is devoted to the study of generalities on variation of mixed Hodge structure. In Section 4, we discuss the variations of mixed Hodge structures arising from the mixed Hodge structures on compact support cohomology groups of quasi-projective varieties. We show that they are graded polarizable and admissible. Section 5 is the main part of this paper. Here, we characterize higher direct images of log canonical divisors by using canonical extensions of Hodge bundles (cf. Theorem 5.1 and Theorem 5.3). It is a generalization of the results by Yujiro Kawamata, Noboru Nakayama, János Kollár, Morihiro Saito, and Osamu Fujino. Note that some technical parts will be treated in subsequent sections. In Section 6, we discuss a purely Hodge theoretic aspect of the Fujita–Kawamata semi-positivity theorem. Our formulation is different from Kawamata’s original one but is suited for our results in Section 5. In Section 7, we discuss some generalizations of vanishing and torsion-free theorems for quasi-projective simple normal crossing pairs. They are necessary for the arguments in Section 5. In Section 8, we treat some examples, which help us understand the Fujita–Kawamata semi-positivity theorem, Viehweg’s weak positivity theorem, and so on, in details.

Let us recall basic definitions and notation.

Notation. For a proper morphism $f : X \rightarrow Y$, the *exceptional locus*, which is denoted by $\text{Exc}(f)$, is the locus where f is not an isomorphism. \mathbb{R} (resp. \mathbb{Q}) denotes the set of real (resp. rational) numbers. \mathbb{Z} denotes the set of integers.

1.9 (Divisors, \mathbb{Q} -divisors, and \mathbb{R} -divisors). For an \mathbb{R} -Weil divisor $D = \sum_{j=1}^r d_j D_j$ on a normal variety X such that D_i is a prime divisor for every i and that $D_i \neq D_j$ for $i \neq j$, we define the *round-up* $\lceil D \rceil = \sum_{j=1}^r \lceil d_j \rceil D_j$ (resp. the *round-down* $\lfloor D \rfloor = \sum_{j=1}^r \lfloor d_j \rfloor D_j$), where for any real number x , $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \leq \lceil x \rceil < x + 1$ (resp. $x - 1 < \lfloor x \rfloor \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \lfloor D \rfloor$. We call D a *boundary* (resp. *subboundary*) \mathbb{R} -divisor if $0 \leq d_j \leq 1$ (resp. $d_j \leq 1$) for every j . \mathbb{Q} -linear equivalence (resp. \mathbb{R} -linear equivalence) of two \mathbb{Q} -divisors (resp. \mathbb{R} -divisors) B_1 and B_2 is denoted by $B_1 \sim_{\mathbb{Q}} B_2$ (resp. $B_1 \sim_{\mathbb{R}} B_2$).

1.10 (Singularities of pairs). Let X be a normal variety and let Δ be an effective \mathbb{R} -divisor on X such that $K_X + \Delta$ is \mathbb{R} -Cartier. Let $f : X \rightarrow Y$

be a resolution such that $\text{Exc}(f) \cup f_*^{-1}\Delta$ has a simple normal crossing support, where $f_*^{-1}\Delta$ is the strict transform of Δ on Y . We can write

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i.$$

We say that (X, Δ) is *log canonical* (*lc*, for short) if $a_i \geq -1$ for every i . We usually write $a_i = a(E_i, X, \Delta)$ and call it the *discrepancy coefficient* of E with respect to (X, Δ) .

If (X, Δ) is log canonical and there exist a resolution $f : Y \rightarrow X$ and a divisor E on Y such that $a(E, X, \Delta) = -1$, then $f(E)$ is called a *log canonical center* (*lc center*, for short) with respect to (X, Δ) .

It is very important to understand the following example.

1.11 (A basic example). Let X be a smooth variety and let Δ be a reduced simple normal crossing divisor on X . Then the pair (X, Δ) is log canonical. Let $\Delta = \sum_{i \in I} \Delta_i$ be the irreducible decomposition of Δ . Then a subvariety W of X is a log canonical center with respect to (X, Δ) if and only if W is an irreducible component of $\Delta_{i_1} \cap \cdots \cap \Delta_{i_k}$ for some $\{i_1, \dots, i_k\} \subset I$.

We will work over \mathbb{C} , the complex number field, throughout this paper.

Acknowledgments. The first author was partially supported by The Inamori Foundation and by the Grant-in-Aid for Young Scientists (A) #20684001 from JSPS. He would like to thank Professors Takeshi Abe, Hiraku Kawanoue, Kenji Matsuki, and Shigefumi Mori for discussions. He also thanks Professors Valery Alexeev and Christopher Hacon for introducing him this problem. He thanks Professor Gregory James Pearlstein (cf. [P]) and Professor Fouad El Zein for answering his questions and giving him some useful comments. He thanks Professor Kazuya Kato for fruitful discussions. The authors would like to thank Professors Kazuya Kato, Chikara Nakayama, and Sampei Usui for discussions and comments.

2. PRELIMINARIES

This section collects some preliminary facts. First, we quickly recall basic definitions of divisors. We note that we have to deal with reducible algebraic schemes in this paper. For details, see [Mu1, Lecture 9] and [Fu, Appendix B.4].

2.1. Let X be a noetherian scheme with structure sheaf \mathcal{O}_X and let \mathcal{K}_X be the sheaf of total quotient rings of \mathcal{O}_X , that is, for every affine

open set $U \subset X$, $\Gamma(U, \mathcal{K}_X)$ is the total quotient ring of $\Gamma(U, \mathcal{O}_X)$. Let \mathcal{K}_X^* denote the (multiplicative) sheaf of invertible elements in \mathcal{K}_X , and \mathcal{O}_X^* the sheaf of invertible elements in \mathcal{O}_X . We note that $\mathcal{O}_X \subset \mathcal{K}_X$ and $\mathcal{O}_X^* \subset \mathcal{K}_X^*$.

2.2 (Cartier, \mathbb{Q} -Cartier, and \mathbb{R} -Cartier divisors). A *Cartier divisor* D on X is a global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$, that is, D is an element of $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. A *\mathbb{Q} -Cartier \mathbb{Q} -divisor* (resp. *\mathbb{R} -Cartier \mathbb{R} -divisor*) is an element of $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \otimes_{\mathbb{Z}} \mathbb{R}$).

2.3 (Linear, \mathbb{Q} -linear, and \mathbb{R} -linear equivalence). Let D_1 and D_2 be two \mathbb{R} -Cartier \mathbb{R} -divisors on X . Then D_1 is *linearly* (resp. *\mathbb{Q} -linearly*, or *\mathbb{R} -linearly*) *equivalent* to D_2 , denoted by $D_1 \sim D_2$ (resp. $D_1 \sim_{\mathbb{Q}} D_2$, or $D_1 \sim_{\mathbb{R}} D_2$) if

$$D_1 = D_2 + \sum_{i=1}^k r_i(f_i)$$

such that $f_i \in \Gamma(X, \mathcal{K}_X^*)$ and $r_i \in \mathbb{Z}$ (resp. $r_i \in \mathbb{Q}$, or $r_i \in \mathbb{R}$) for every i . We note that (f_i) is a *principal Cartier divisor* associated to f_i , that is, the image of f_i by $\Gamma(X, \mathcal{K}_X^*) \rightarrow \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. Let $f : X \rightarrow Y$ be a morphism. If there is an \mathbb{R} -Cartier \mathbb{R} -divisor B on Y such that $D_1 \sim_{\mathbb{R}} D_2 + f^*B$, then D_1 is said to be *relatively \mathbb{R} -linearly equivalent* to D_2 . It is denoted by $D_1 \sim_{\mathbb{R}, f} D_2$.

2.4 (Supports). Let D be a Cartier divisor on X . The *support* of D , denoted by $\text{Supp} D$, is the subset of X consisting of points x such that a local equation for D is not in $\mathcal{O}_{X,x}^*$. The support of D is a closed subset of X .

2.5 (Weil divisors, \mathbb{Q} -divisors, and \mathbb{R} -divisors). Let X be an equidimensional reduced separated algebraic scheme. We note that X is not necessarily regular in codimension one. A (*Weil*) *divisor* D on X is a finite formal sum

$$\sum_{i=1}^n d_i D_i$$

where D_i is an irreducible reduced closed subscheme of X of pure codimension one and d_i is an integer for every i such that $D_i \neq D_j$ for $i \neq j$.

If $d_i \in \mathbb{Q}$ (resp. $d_i \in \mathbb{R}$) for every i , then D is called a *\mathbb{Q} -divisor* (resp. *\mathbb{R} -divisor*). We define the *round-up* $\lceil D \rceil = \sum_{i=1}^r \lceil d_i \rceil D_i$ (resp. the *round-down* $\lfloor D \rfloor = \sum_{i=1}^r \lfloor d_i \rfloor D_i$), where for every real number x , $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \leq \lceil x \rceil < x + 1$ (resp. $x - 1 < \lfloor x \rfloor \leq x$). The *fractional part* $\{D\}$ of D denotes $D - \lfloor D \rfloor$. We

define $D^{<1} = \sum_{d_i < 1} d_i D_i$ and so on. We call D a *boundary* \mathbb{R} -divisor if $0 \leq d_i \leq 1$ for every i .

Next, we recall the definition of *simple normal crossing pairs*.

Definition 2.6 (Simple normal crossing pairs). We say that the pair (X, D) is *simple normal crossing* at a point $a \in X$ if X has a Zariski open neighborhood U of a that can be embedded in a smooth variety Y , where Y has regular local coordinates $(x_1, \dots, x_p, y_1, \dots, y_r)$ at $a = 0$ in which U is defined by a monomial equation

$$x_1 \cdots x_p = 0$$

and

$$D = \sum_{i=1}^r \alpha_i (y_i = 0)|_U, \quad \alpha_i \in \mathbb{R}.$$

We say that (X, D) is a *simple normal crossing pair* if it is simple normal crossing at every point of X . If $(X, 0)$ is a simple normal crossing pair, then X is called a *simple normal crossing variety*. If (X, D) is a simple normal crossing pair, then X has only Gorenstein singularities. Thus, it has an invertible dualizing sheaf ω_X . Therefore, we can define the *canonical divisor* K_X such that $\omega_X \simeq \mathcal{O}_X(K_X)$ (cf. [Go, Proposition (21.3.4)]). It is a Cartier divisor on X and is well-defined up to linear equivalence.

We say that a simple normal crossing pair (X, D) is *embedded* if there exists a closed embedding $\iota : X \rightarrow M$, where M is a smooth variety of dimension $\dim X + 1$.

We note that a simple normal crossing pair is called a *semi-snc pair* in [Ko5, Definition 1.9].

Definition 2.7 (Strata and permissibility). Let X be a simple normal crossing variety and let $X = \bigcup_{i \in I} X_i$ be the irreducible decomposition of X . A *stratum* of X is an irreducible component of $X_{i_1} \cap \cdots \cap X_{i_k}$ for some $\{i_1, \dots, i_k\} \subset I$. A Cartier divisor D on X is *permissible* if D contains no strata of X in its support. A finite \mathbb{Q} -linear (resp. \mathbb{R} -linear) combination of permissible Cartier divisors is called a *permissible \mathbb{Q} -divisor* (resp. *\mathbb{R} -divisor*) on X .

2.8. Let X be a simple normal crossing variety. Let $\text{PerDiv}(X)$ be the abelian group generated by permissible Cartier divisors on X and let $\text{Weil}(X)$ be the abelian group generated by Weil divisors on X . Then we can define natural injective homomorphisms of abelian groups

$$\psi : \text{PerDiv}(X) \otimes_{\mathbb{Z}} \mathbb{K} \rightarrow \text{Weil}(X) \otimes_{\mathbb{Z}} \mathbb{K}$$

for $\mathbb{K} = \mathbb{Z}, \mathbb{Q}$, and \mathbb{R} . Let $\nu : \tilde{X} \rightarrow X$ be the normalization. Then we have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Div}(\tilde{X}) \otimes_{\mathbb{Z}} \mathbb{K} & \xrightarrow[\tilde{\psi}]{\sim} & \mathrm{Weil}(\tilde{X}) \otimes_{\mathbb{Z}} \mathbb{K} \\ \nu^* \uparrow & & \downarrow \nu_* \\ \mathrm{PerDiv}(X) \otimes_{\mathbb{Z}} \mathbb{K} & \xrightarrow[\psi]{} & \mathrm{Weil}(X) \otimes_{\mathbb{Z}} \mathbb{K} \end{array}$$

Note that $\mathrm{Div}(\tilde{X})$ is the abelian group generated by Cartier divisors on \tilde{X} and that $\tilde{\psi}$ is an isomorphism since \tilde{X} is smooth.

By the injection ψ , every permissible Cartier (resp. \mathbb{Q} -Cartier or \mathbb{R} -Cartier) divisor can be considered as a Weil divisor (resp. \mathbb{Q} -divisor or \mathbb{R} -divisor). Therefore, various operations, for example, $\lfloor D \rfloor$, $D^{<1}$, and so on, make sense for a permissible \mathbb{R} -Cartier \mathbb{R} -divisor D on X .

We note the following easy example.

Example 2.9. Let X be a simple normal crossing variety in $\mathbb{C}^3 = \mathrm{Spec} \mathbb{C}[x, y, z]$ defined by $xy = 0$. We put $D_1 = (x + z = 0) \cap X$ and $D_2 = (x - z = 0) \cap X$. Then $D = \frac{1}{2}D_1 + \frac{1}{2}D_2$ is a permissible \mathbb{Q} -Cartier \mathbb{Q} -divisor on X . In this case, $\lfloor D \rfloor = (x = z = 0)$ on X . Therefore, $\lfloor D \rfloor$ is not a Cartier divisor on X .

Definition 2.10 (Simple normal crossing divisors). Let X be a simple normal crossing variety and let D be a permissible Cartier divisor on X . If D is reduced and (X, D) is a simple normal crossing pair, then D is called a *simple normal crossing divisor* on X .

Remark 2.11. Let X be a simple normal crossing variety and let D be a permissible \mathbb{K} -Cartier \mathbb{K} -divisor on X where $\mathbb{K} = \mathbb{Q}$ or \mathbb{R} . If $\mathrm{Supp} D$ is a simple normal crossing divisor on X , then $\lfloor D \rfloor$ and $\lceil D \rceil$ (resp. $\{D\}$, $D^{<1}$, and so on) are Cartier (resp. \mathbb{K} -Cartier) divisors on X .

Definition 2.12 (Strata and permissibility for pairs). Let (X, D) be a simple normal crossing pair such that D is a boundary \mathbb{R} -divisor on X . Let $\nu : X^\nu \rightarrow X$ be the normalization. We define Θ by the formula

$$K_{X^\nu} + \Theta = \nu^*(K_X + D).$$

Then a *stratum* of (X, D) is an irreducible component of X or the ν -image of a log canonical center of (X^ν, Θ) . We note that (X^ν, Θ) is log canonical (see 1.11). When $D = 0$, this definition is compatible with Definition 2.7. A permissible \mathbb{R} -Cartier \mathbb{R} -divisor B on X is *permissible with respect to (X, D)* if B contains no strata of (X, D) in its support.

The reader will find that it is very useful to introduce the notion of *globally embedded simple normal crossing pairs* for the proof of vanishing and torsion-free theorems (cf. [F7, Chapter 2]).

Definition 2.13 (Globally embedded simple normal crossing pairs). Let X be a simple normal crossing divisor on a smooth variety M and let B be an \mathbb{R} -divisor on M such that $\text{Supp}(B + X)$ is a simple normal crossing divisor and that B and X have no common irreducible components. We put $D = B|_X$ and consider the pair (X, D) . We call (X, D) a *globally embedded simple normal crossing pair*. In this case, it is obvious that (X, D) is an embedded simple normal crossing pair (cf. Definition 2.6).

In Section 7, we will discuss some vanishing and torsion-free theorems for *quasi-projective* simple normal crossing pairs, which will play crucial roles in Section 5. See also [F14] and [F15].

3. GENERALITIES ON VARIATION OF MIXED HODGE STRUCTURE

3.1. Let X be a complex analytic variety. For a point $x \in X$, we denote by $\mathbb{C}(x) (\simeq \mathbb{C})$ the residue field at the point x . For a morphism $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ of \mathcal{O}_X -modules we denote the morphism

$$\varphi \otimes \text{id} : \mathcal{F} \otimes \mathbb{C}(x) \rightarrow \mathcal{G} \otimes \mathbb{C}(x)$$

by $\varphi(x)$ for a point $x \in X$.

Remark 3.2. On a complex manifold X , the sheaf $\mathbb{C}(x)$ has a finite Tor dimension for all $x \in X$. Therefore the (bi)filtered derived tensor product $\otimes^L \mathbb{C}(x)$ makes sense for any complex of \mathcal{O}_X -modules equipped with a finite decreasing filtration (and a finite increasing filtration).

Remark 3.3. For a complex K equipped with a finite decreasing filtration F and for an integer q , the four conditions

$$(3.3.1) \quad d : K^q \rightarrow K^{q+1} \text{ is strictly compatible with the filtration } F,$$

$$(3.3.2) \quad \text{the canonical morphism } H^{q+1}(F^p K) \rightarrow H^{q+1}(K) \text{ is injective for all } p,$$

$$(3.3.3) \quad \text{the canonical morphism } H^{q+1}(F^{p+1} K) \rightarrow H^{q+1}(F^p K) \text{ is injective for all } p,$$

$$(3.3.4) \quad \text{the canonical morphism } H^q(F^p K) \rightarrow H^q(\text{Gr}_F^p K) \text{ is surjective for all } p$$

are equivalent. Therefore the strict compatibility in (3.3.1) makes sense in the bifiltered derived category.

The following definition is an analogue of the notion of perfect complex (see e.g. [FGAE, 8.3.6.3]).

Definition 3.4. Let X be a complex variety. A complex of \mathcal{O}_X -modules K equipped with a finite decreasing filtration F (resp. finite decreasing filtrations F, G) is called a filtered perfect (resp. bifiltered perfect) complex if $\mathrm{Gr}_F^p K$ (resp. $\mathrm{Gr}_F^p \mathrm{Gr}_G^q K$) is a perfect complex for all p (resp. for all p, q). For the case of increasing filtration, we use the similar notation.

We frequently use the following lemma throughout this section.

Lemma 3.5. *Let X be a complex manifold.*

(i) *For a perfect complex K on X , the function*

$$X \ni x \mapsto \dim H^q(K \otimes \mathbb{C}(x))$$

is upper semi-continuous for all q .

(ii) *Let K be a perfect complex on X . If there exists an integer q_0 such that $H^q(K)$ is locally free of finite rank for all $q \geq q_0$, then the canonical morphism*

$$H^q(K) \otimes \mathcal{F} \longrightarrow H^q(K \otimes^L \mathcal{F})$$

is an isomorphism for any \mathcal{O}_X -module \mathcal{F} and for all $q \geq q_0$.

(iii) *Fix an integer q . For a perfect complex K on X , the following two conditions are equivalent:*

(3.5.1) *The function*

$$X \ni x \mapsto \dim H^q(K \otimes^L \mathbb{C}(x))$$

is locally constant.

(3.5.2) *The sheaf $H^q(K)$ is locally free of finite rank and the canonical morphism*

$$H^q(K) \otimes \mathcal{F} \longrightarrow H^q(K \otimes^L \mathcal{F})$$

is an isomorphism for any \mathcal{O}_X -module \mathcal{F} .

Moreover, if these equivalent conditions are satisfied, then the canonical morphism

$$H^{q-1}(K) \otimes \mathcal{F} \longrightarrow H^{q-1}(K \otimes^L \mathcal{F})$$

is an isomorphism for any \mathcal{O}_X -module \mathcal{F} .

(iv) *Let (K, F) be a filtered perfect complex on X . Assume that the function*

$$X \ni x \mapsto \dim H^q(K \otimes^L \mathbb{C}(x))$$

is locally constant. If the morphisms

$$\begin{aligned} d(x) : (K \otimes^L \mathbb{C}(x))^{q-1} &\longrightarrow (K \otimes^L \mathbb{C}(x))^q \\ d(x) : (K \otimes^L \mathbb{C}(x))^q &\longrightarrow (K \otimes^L \mathbb{C}(x))^{q+1} \end{aligned}$$

are strictly compatible with the filtration $F(K \otimes^L \mathbb{C}(x))$ for every $x \in X$, then $H^q(\mathrm{Gr}_F^p K)$ is locally free of finite rank, the canonical morphism

$$H^q(\mathrm{Gr}_F^p K) \otimes \mathbb{C}(x) \simeq H^q(\mathrm{Gr}_F^p (K \otimes^L \mathbb{C}(x))) \quad (3.5.3)$$

is an isomorphism for all p and for every $x \in X$, and $d : K^q \longrightarrow K^{q+1}$ is strictly compatible with the filtration F .

Proof. We can easily obtain (i), (ii) and (iii) by the arguments in [Mu2, Chapter 5].

The strict compatibility conditions in (iv) imply the exactness of the sequence

$$\begin{aligned} 0 \longrightarrow H^q(F^{p+1}(K \otimes^L \mathbb{C}(x))) &\longrightarrow H^q(F^p(K \otimes^L \mathbb{C}(x))) \\ &\longrightarrow H^q(\mathrm{Gr}_F^p K \otimes^L \mathbb{C}(x)) \longrightarrow 0 \end{aligned}$$

for all p and for every $x \in X$. Thus we obtain the equality

$$\sum_p \dim H^q(\mathrm{Gr}_F^p K \otimes^L \mathbb{C}(x)) = \dim H^q(K \otimes^L \mathbb{C}(x))$$

for every x , which implies that $\dim H^q(\mathrm{Gr}_F^p K \otimes^L \mathbb{C}(x))$ is locally constant with respect to $x \in X$. Applying (iii), $H^q(\mathrm{Gr}_F^p K)$ is locally free and (3.5.3) is an isomorphism for all p and for any $x \in X$. By using the isomorphisms (3.5.3) for all p , we can easily see the surjectivity of the canonical morphism

$$H^q(F^p K) \otimes \mathbb{C}(x) \longrightarrow H^q(\mathrm{Gr}_F^p K) \otimes \mathbb{C}(x)$$

for any $x \in X$, and then the canonical morphism

$$H^q(F^p K) \longrightarrow H^q(\mathrm{Gr}_F^p K)$$

is surjective for every p . Thus the morphism $d : K^q \longrightarrow K^{q+1}$ is strictly compatible with the filtration F by Remark 3.3. \square

Next, we define new notions for the later use.

Definition 3.6. Let X be a complex manifold. A pre-variation of \mathbb{Q} -Hodge structure of weight m on X is a triple $V = (\mathbb{V}, (\mathcal{V}, F), \alpha)$ such that

- \mathbb{V} is a local system of finite dimensional \mathbb{Q} -vector space on X ,
- \mathcal{V} is an \mathcal{O}_X -module and F is a finite decreasing filtration on \mathcal{V} ,
- $\alpha : \mathbb{V} \longrightarrow \mathcal{V}$ is a morphism of \mathbb{Q} -sheaves,

satisfying the conditions

(3.6.1) α induces an isomorphism $\mathcal{O}_X \otimes \mathbb{V} \simeq \mathcal{V}$ of \mathcal{O}_X -modules,

(3.6.2) $\mathrm{Gr}_F^p \mathcal{V}$ is a locally free \mathcal{O}_X -module of finite rank for every p ,

(3.6.3) $(\mathbb{V}_x, F(\mathcal{V}(x)))$ is a Hodge structure of weight m for every $x \in X$, where we identify $\mathbb{V}_x \otimes \mathbb{C}$ with $\mathcal{V}(x)$ by the isomorphism $\alpha(x)$.

We denote $(\mathbb{V}_x, F(\mathcal{V}(x)))$ by $V(x)$ for $x \in X$.

We identify $\mathcal{O}_X \otimes \mathbb{V}$ with \mathcal{V} by the isomorphism in (3.6.1) if there is no danger of confusion. Under this identification, we write $V = (\mathbb{V}, F)$ for a pre-variation of \mathbb{Q} -Hodge structure.

We define the notion of a morphism of pre-variations in the trivial way.

Remark 3.7. A variation of \mathbb{Q} -Hodge structure of weight m on X is nothing but a pre-variation $V = (\mathbb{V}, F)$ of \mathbb{Q} -Hodge structure of weight m , such that the canonical integrable connection ∇ on $\mathcal{V} = \mathcal{O}_X \otimes \mathbb{V}$ satisfies the Griffiths transversality

$$\nabla(F^p) \subset \Omega_X^1 \otimes F^{p-1} \quad (3.7.1)$$

for every p . A morphism of variations of \mathbb{Q} -Hodge structures is a morphism of underlying pre-variations of \mathbb{Q} -Hodge structures.

Remark 3.8. (i) Let $V_1 = (\mathbb{V}_1, F)$ and $V_2 = (\mathbb{V}_2, F)$ be pre-variations of \mathbb{Q} -Hodge structures of weight m_1 and m_2 respectively. Then the local systems $\mathbb{V}_1 \otimes \mathbb{V}_2$ and $\mathcal{H}om(\mathbb{V}_1, \mathbb{V}_2)$ underlie pre-variations of \mathbb{Q} -Hodge structures of weight $m_1 + m_2$ and $m_2 - m_1$ respectively. We denote these pre-variations of \mathbb{Q} -Hodge structures by $V_1 \otimes V_2$ and $\mathcal{H}om(V_1, V_2)$ respectively.

(ii) For an integer n , $\mathbb{Q}_X(n)$ denotes the pre-variation of \mathbb{Q} -Hodge structure of Tate as usual. This is, in fact, a variation of \mathbb{Q} -Hodge structure of weight $-2n$ on X . For a pre-variation V of \mathbb{Q} -Hodge structure of weight m , $V(n) = V \otimes \mathbb{Q}_X(n)$ is a pre-variation of \mathbb{Q} -Hodge structure of weight $m - 2n$, which is called the Tate twist of V as usual.

Definition 3.9. Let X be a complex manifold and $V = (\mathbb{V}, F)$ a pre-variation of \mathbb{Q} -Hodge structure of weight m on X . A polarization on V is a morphism of pre-variations of \mathbb{Q} -Hodge structures

$$V \otimes V \longrightarrow \mathbb{Q}_X(-m)$$

which induces a polarization on $V(x)$ for every point $x \in X$. A pre-variation of \mathbb{Q} -Hodge structure of weight m is said to be polarizable,

if there exists a polarization on it. A morphism of polarizable pre-variations of \mathbb{Q} -Hodge structures is a morphism of the underlying pre-variations of \mathbb{Q} -Hodge structures.

Definition 3.10. Let X be a complex manifold.

(i) A pre-variation of \mathbb{Q} -mixed Hodge structure on X is a triple $V = ((\mathbb{V}, W), (\mathcal{V}, W, F), \alpha)$ consisting of

- a local system of finite dimensional \mathbb{Q} -vector space \mathbb{V} , equipped with a finite increasing filtration W by local subsystems,
- an \mathcal{O}_X -module \mathcal{V} equipped with a finite increasing filtration W and a finite decreasing filtration F ,
- a morphism of \mathbb{Q} -sheaves $\alpha : \mathbb{V} \longrightarrow \mathcal{V}$ preserving the filtration W

such that the triple $\mathrm{Gr}_m^W V = (\mathrm{Gr}_m^W \mathbb{V}, (\mathrm{Gr}_m^W \mathcal{V}, F), \mathrm{Gr}_m^W \alpha)$ is a pre-variation of \mathbb{Q} -Hodge structure of weight m for every m .

We identify $(\mathcal{O}_X \otimes \mathbb{V}, W)$ and (\mathcal{V}, W) by the isomorphism induced by α as before, if there is no danger of confusion. Under this identification, we use the notation $V = (\mathbb{V}, W, F)$ for a pre-variation of \mathbb{Q} -mixed Hodge structure.

(ii) A pre-variation $V = (\mathbb{V}, W, F)$ of \mathbb{Q} -mixed Hodge structure on X is called *graded polarizable*, if $\mathrm{Gr}_m^W V$ is a polarizable pre-variation of \mathbb{Q} -Hodge structure for every m .

(iii) We define the notion of a morphism of pre-variations of \mathbb{Q} -mixed Hodge structures by the trivial way. A morphism of polarizable pre-variations of \mathbb{Q} -mixed Hodge structures is a morphism of the underlying pre-variations.

Now, let us recall the definition of *graded polarizable variation of \mathbb{Q} -mixed Hodge structure* (GPVMHS, for short). See, for example, [SZ, §3], [SSU, Part I, Section 1], [BZ, Section 7], [PS, Definitions 14.44 and 14.45], and so on.

Definition 3.11 (GPVMHS). Let X be a complex manifold.

(i) A pre-variation $V = (\mathbb{V}, W, F)$ of \mathbb{Q} -mixed Hodge structure on X is said to be a variation of \mathbb{Q} -mixed Hodge structure, if the canonical integrable connection ∇ on $\mathcal{V} \simeq \mathcal{O}_X \otimes \mathbb{V}$ satisfies the Griffiths transversality (3.7.1) in Remark 3.7.

(ii) A variation of \mathbb{Q} -Hodge structure is called *graded polarizable*, if the underlying pre-variation is graded polarizable.

(iii) A morphism of (graded polarizable) variations of \mathbb{Q} -mixed Hodge structures is a morphism of the underlying pre-variations.

The following definition of the *admissibility* is given by Steenbrink–Zucker [SZ, (3.13) Properties] in the one-dimensional case and by Kashiwara [Ks, 1.8, 1.9] in the general case [Ks, §1]. See also [PS, Definition 14.49].

Definition 3.12 (Admissibility). (i) A variation of \mathbb{Q} -mixed Hodge structure $V = (\mathbb{V}, W, F)$ over $\Delta^* = \Delta \setminus \{0\}$, where $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$, is said to be *pre-admissible* if it satisfies:

(3.12.1) The monodromy around the origin is quasi-unipotent.

(3.12.2) Let $\tilde{\mathcal{V}}$ and $W_k \tilde{\mathcal{V}}$ be the upper canonical extensions of $\mathcal{V} = \mathcal{O}_{\Delta^*} \otimes \mathbb{V}$ and of $\mathcal{O}_{\Delta^*} \otimes W_k \mathbb{V}$ in the sense of Deligne [D1, Remarques 5.5 (i)] (see also [Ko2, Section 2]). Then the filtration F on \mathcal{V} extends to the filtration F on $\tilde{\mathcal{V}}$ such that $\mathrm{Gr}_F^p \mathrm{Gr}_k^W \tilde{\mathcal{V}}$ is locally free \mathcal{O}_{Δ} -modules of finite rank for each k, p .

(3.12.3) The logarithm of the unipotent part of the monodromy admits a weight filtration relative to W .

(ii) Let X be a complex variety and U a nonsingular Zariski open subset of X . A variation of \mathbb{Q} -mixed Hodge structure V on U is said to be *admissible* (with respect to X) if for every morphism $i : \Delta \rightarrow X$ with $i(\Delta^*) \subset U$, the variation i^*V on Δ^* is pre-admissible.

We frequently use the following lemma in Section 6, which is a special case of [Ks, Proposition 1.11.3].

Proposition 3.13. *Let X be a complex manifold, U the complement of a normal crossing divisor on X and $V = (\mathbb{V}, W, F)$ a variation of \mathbb{R} -mixed Hodge structure on U . We denote the upper canonical extensions of $\mathcal{V} = \mathcal{O}_U \otimes \mathbb{V}$ and of $W_k \mathcal{V} = \mathcal{O}_U \otimes W_k \mathbb{V}$ by $\tilde{\mathcal{V}}$ and by $W_k \tilde{\mathcal{V}}$ respectively. If V is admissible on U with respect to X , then the filtration F on \mathcal{V} extends to a finite filtration F on $\tilde{\mathcal{V}}$ by subbundles such that $\mathrm{Gr}_F^p \mathrm{Gr}_k^W \tilde{\mathcal{V}}$ is a locally free \mathcal{O}_X -module of finite rank for all k, p .*

We give an elementary but useful remark on the quasi-unipotency of monodromy.

Remark 3.14 (Quasi-unipotency). If the local system \mathbb{V} has a \mathbb{Z} -structure, that is, there is a local system $\mathbb{V}_{\mathbb{Z}}$ on X of \mathbb{Z} -modules of finite rank such that $\mathbb{V} = \mathbb{V}_{\mathbb{Z}} \otimes \mathbb{Q}$, in Definition 3.12, then the quasi-unipotency automatically follows from Borel’s theorem (cf. [Sc, (4.5) Lemma (Borel)]).

The following lemma states the fundamental results on pre-variations of \mathbb{Q} -Hodge structures.

Lemma 3.15. *Let X be a complex manifold.*

(i) *The category of the pre-variations of \mathbb{Q} -Hodge structures of weight m on X is an abelian category for every m .*

(ii) *Let V_1 and V_2 be pre-variations of \mathbb{Q} -Hodge structures of weight m_1 and m_2 respectively, and $\varphi : V_1 \rightarrow V_2$ a morphism of pre-variations. If $m_1 > m_2$, then $\varphi = 0$.*

(iii) *Let $\varphi : V_1 \rightarrow V_2$ be a morphism of pre-variations $V_1 = (\mathbb{V}_1, F)$ and $V_2 = (\mathbb{V}_2, F)$ of \mathbb{Q} -Hodge structures of weight m on X . Then the induced morphism $\varphi \otimes \text{id} : \mathbb{V}_1 \otimes \mathcal{O}_X \rightarrow \mathbb{V}_2 \otimes \mathcal{O}_X$ is strictly compatible with the filtration F .*

(iv) *The functor from the category of the pre-variations of \mathbb{Q} -Hodge structures of weight m to the category of the \mathbb{Q} -Hodge structure of weight m which assigns V to $V(x)$ is an exact functor for every $x \in X$.*

(v) *The category of the polarizable variations of \mathbb{Q} -Hodge structures of weight m on X is an abelian category for every m .*

Proof. The statements (i), (iii) and (iv) are easy consequences of Lemma 3.5 (iv), and (ii) is easily proved by the corresponding result for \mathbb{Q} -Hodge structures. So we prove (v) now.

Let $V_1 = (\mathbb{V}_1, F)$ and $V_2 = (\mathbb{V}_2, F)$ be polarizable pre-variations of \mathbb{Q} -Hodge structures of weight m on X , and $\varphi : V_1 \rightarrow V_2$ a morphism. We fix polarizations on V_1 and V_2 respectively. Taking (i) into the account, it is sufficient to prove that $\text{Ker}(\varphi)$ and $\text{Coker}(\varphi)$ are polarizable. The case of $\text{Ker}(\varphi)$ is trivial. Then we discuss the case of $\text{Coker}(\varphi)$.

The morphism φ induces a morphism

$$\varphi^* : \mathcal{H}om(V_2, \mathbb{Q}_X(-m)) \rightarrow \mathcal{H}om(V_1, \mathbb{Q}_X(-m))$$

which is clearly a morphism of pre-variations of \mathbb{Q} -Hodge structures of weight m . On the other hand, the polarizations on V_1 and V_2 induce the identifications

$$V_1 \simeq \mathcal{H}om(V_1, \mathbb{Q}_X(-m)), \quad V_2 \simeq \mathcal{H}om(V_2, \mathbb{Q}_X(-m))$$

which are isomorphisms of pre-variations of \mathbb{Q} -Hodge structures. By these identifications the morphism φ^* above can be considered as a morphism of pre-variations $V_2 \rightarrow V_1$, which we denote by the same symbol φ^* by abuse of the language. Then the inclusion $\text{Ker}(\varphi^*) \hookrightarrow V_2$ induces an isomorphism $\text{Ker}(\varphi^*) \simeq \text{Coker}(\varphi)$ of pre-variations. Therefore we obtain a polarization expected. \square

Definition 3.16. Let X be a complex manifold.

(i) Let $K = (K_{\mathbb{Q}}, (K_{\mathcal{O}}, F), \alpha)$ be a triple consisting of

- a complex $K_{\mathbb{Q}}$ of \mathbb{Q} -sheaves on X bounded below,

- a complex $K_{\mathcal{O}}$ of \mathcal{O}_X -sheaves on X bounded below, with a finite decreasing filtration F ,
- a morphism $\alpha : K_{\mathbb{Q}} \longrightarrow K_{\mathcal{O}}$ of complexes of \mathbb{Q} -sheaves.

The data K is called a pre-variation of \mathbb{Q} -Hodge complex of weight m , if the following conditions hold:

(3.16.1) The triple

$$H^n(K) = (H^n(K_{\mathbb{Q}}), (H^n(K_{\mathcal{O}}), F), H^n(\alpha))$$

is a pre-variation of \mathbb{Q} -Hodge structure of weight $m + n$ for every integer n .

(3.16.2) The spectral sequence $E_r^{p,q}(K_{\mathcal{O}}, F)$ associated to the filtered complex $(K_{\mathcal{O}}, F)$ degenerates at E_1 -terms.

Moreover, the pre-variation of \mathbb{Q} -Hodge complex K of weight m is said to be polarizable, if $H^n(K)$ is polarizable for every n .

(ii) Let $K = ((K_{\mathbb{Q}}, W), (K_{\mathcal{O}}, W, F), \alpha)$ be a triple consisting of

- a complex $K_{\mathbb{Q}}$ of \mathbb{Q} -sheaves on X bounded below, with a finite increasing filtration W ,
- a complex $K_{\mathcal{O}}$ of \mathcal{O}_X -sheaves on X bounded below, with a finite increasing filtration W and a finite decreasing filtration F ,
- a morphism $\alpha : K_{\mathbb{Q}} \longrightarrow K_{\mathcal{O}}$ of complexes of \mathbb{Q} -sheaves preserving the filtrations W on both sides.

The data K is called a pre-variation of \mathbb{Q} -mixed Hodge complex if the triple

$$\mathrm{Gr}_m^W K = (\mathrm{Gr}_m^W K_{\mathbb{Q}}, (\mathrm{Gr}_m^W K_{\mathcal{O}}, F), \mathrm{Gr}_m^W \alpha)$$

is a pre-variation of \mathbb{Q} -Hodge complex of weight m for every m . Moreover a variation of \mathbb{Q} -mixed Hodge complex K is said to be graded polarizable, if $\mathrm{Gr}_m^W K$ is polarizable for every m .

(iii) Let $K = (K_{\mathbb{Q}}, (K_{\mathcal{O}}, F), \alpha)$ and $K' = (K'_{\mathbb{Q}}, (K'_{\mathcal{O}}, F), \alpha')$ be pre-variations of \mathbb{Q} -Hodge complexes of certain weights. A morphism of pre-variations $\varphi : K \longrightarrow K'$ is a pair $\varphi = (\varphi_{\mathbb{Q}}, \varphi_{\mathcal{O}})$ consisting of

- a morphism of complexes of \mathbb{Q} -sheaves $\varphi_{\mathbb{Q}} : K_{\mathbb{Q}} \longrightarrow K'_{\mathbb{Q}}$
- a morphism of complexes of \mathcal{O}_X -modules $\varphi_{\mathcal{O}} : K_{\mathcal{O}} \longrightarrow K'_{\mathcal{O}}$ preserving the filtration F

such that the diagram

$$\begin{array}{ccc} K_{\mathbb{Q}} & \xrightarrow{\varphi_{\mathbb{Q}}} & K'_{\mathbb{Q}} \\ \alpha \downarrow & & \downarrow \alpha' \\ K_{\mathcal{O}} & \xrightarrow{\varphi_{\mathcal{O}}} & K'_{\mathcal{O}} \end{array}$$

is commutative. A morphism of pre-variations of \mathbb{Q} -mixed Hodge complexes is defined similarly.

(iv) A co-semi-simplicial pre-variation of \mathbb{Q} -(mixed) Hodge complex on X is a covariant functor from the semi-simplicial category Δ_{mon} (cf. [GNPP, Exposé IV(1.1)]) to the category of the pre-variations of \mathbb{Q} -(mixed) Hodge complexes on X as usual.

Remark 3.17. The notion of a pre-variation of \mathbb{Q} -Hodge complex makes sense in the filtered derived category in the following sense.

Let $K = (K_{\mathbb{Q}}, (K_{\mathcal{O}}, F), \alpha)$ and $K' = (K'_{\mathbb{Q}}, (K'_{\mathcal{O}}, F), \alpha')$ be triples on a complex manifold X as in Definition 3.16. Assume that we have a quasi-isomorphism $K_{\mathbb{Q}} \rightarrow K'_{\mathbb{Q}}$ and a filtered quasi-isomorphism $(K_{\mathcal{O}}, F) \rightarrow (K'_{\mathcal{O}}, F)$ which are compatible with the morphisms α and α' . Then K is a pre-variation of \mathbb{Q} -Hodge complex of weight m if and only if so is K' .

Also the notion of a pre-variation of \mathbb{Q} -mixed Hodge complex makes sense in the bifiltered derived category.

Remark 3.18. Let $K = (K_{\mathbb{Q}}, (K_{\mathcal{O}}, F), \alpha)$ be a pre-variation of \mathbb{Q} -Hodge complex of weight m . If $(K_{\mathcal{O}}, F)$ is a filtered perfect complex, then the canonical morphism

$$H^n(\text{Gr}_F^p K) \otimes \mathbb{C}(x) \rightarrow H^n(\text{Gr}_F^p (K \otimes^L \mathbb{C}(x)))$$

is an isomorphism for all n, p and for every $x \in X$ by the fact that $H^n(\text{Gr}_F^p K) \simeq \text{Gr}_F^p H^n(K)$ is locally free of finite rank for all n, p and by Lemma 3.5 (ii). Moreover, we can easily check the E_1 -degeneracy of the spectral sequence associated to the filtered complex $(K_{\mathcal{O}}, F) \otimes^L \mathbb{C}(x)$ for every $x \in X$. Thus we conclude that

$$K \otimes^L \mathbb{C}(x) = (K_{\mathbb{Q}, x}, (K_{\mathcal{O}}, F) \otimes^L \mathbb{C}(x), \alpha(x))$$

is a \mathbb{Q} -Hodge complex of weight m for every $x \in X$. Moreover we have

$$H^n(K) \otimes \mathbb{C}(x) = H^n(K \otimes^L \mathbb{C}(x))$$

as \mathbb{Q} -Hodge structures of weight $m + n$ for every $x \in X$.

The lemma below is an easy consequence of the definition.

Lemma 3.19. *Let X be a complex manifold. For a co-semi-simplicial pre-variation of \mathbb{Q} -mixed Hodge complexes K^\bullet on X , we give filtrations $\delta(W, L)$ and F on the total single complex sK^\bullet by the same way as in [D3, (7.1.6.1), (7.1.7.1)]. Then the triple $(sK^\bullet, \delta(W, L), F)$ is a pre-variation of \mathbb{Q} -mixed Hodge complexes on X . For the case where K^\bullet is a finite co-semi-simplicial pre-variation of \mathbb{Q} -Hodge complexes of weight*

$m, (sK^\bullet, \delta(L)[m], F)$ is a pre-variation of \mathbb{Q} -mixed Hodge complexes on X , where $\delta(L)$ is the finite increasing filtration defined by

$$\delta(L)_m(sK)^n = \bigoplus_{q \geq -m} (K^q)^{n-q} \quad (3.19.1)$$

for all m, n (cf. [D3, (5.1.9.3)]). Moreover, if K^n is graded polarizable for every n , then so is sK^\bullet .

Similarly, for a morphism of pre-variations of \mathbb{Q} -mixed Hodge complexes $\varphi : K_1 \rightarrow K_2$ on X , the mixed cone $C_M(\varphi)$ defined in [E2, 3.3] is a pre-variation of \mathbb{Q} -mixed Hodge complex on X . If K_1 and K_2 are graded polarizable, then so is $C_M(\varphi)$.

Lemma 3.20. *Let X be a complex manifold and*

$$K = ((K_{\mathbb{Q}}, W), (K_{\mathcal{O}}, W, F), \alpha)$$

a pre-variation of \mathbb{Q} -mixed Hodge complex on X . Then the following holds:

(3.20.1) *the triple*

$$H^n(K) = ((H^n(K_{\mathbb{Q}}), W[n]), (H^n(K_{\mathcal{O}}), W[n], F), H^n(\alpha))$$

is a pre-variation of \mathbb{Q} -mixed Hodge structure for every n ,

(3.20.2) *the spectral sequences $E_r^{p,q}(K_{\mathbb{Q}}, W)$ and $E_r^{p,q}(K_{\mathcal{O}}, W)$ degenerate at E_2 -terms,*

(3.20.3) *the spectral sequence $E_r^{p,q}(K_{\mathcal{O}}, F)$ degenerates at E_1 -terms.*

Moreover, if K is graded polarizable, then $H^n(K)$ is a graded polarizable pre-variation of \mathbb{Q} -mixed Hodge structure for every n .

Proof. We consider the spectral sequences $E_r^{p,q}(K_{\mathbb{Q}}, W), E_r^{p,q}(K_{\mathcal{O}}, W)$ associated to the filtration W . The morphism α induces an isomorphism

$$E_r^{p,q}(\alpha) : E_r^{p,q}(K_{\mathbb{Q}}, W) \otimes \mathcal{O}_X \simeq E_r^{p,q}(K_{\mathcal{O}}, W)$$

for all p, q and for all $r \geq 1$. The filtration F_{rec} on $E_r^{p,q}(K_{\mathcal{O}}, W)$ is defined in [D2, (1.3.11)]. We set

$$E_r^{p,q}(K, W) = (E_r^{p,q}(K_{\mathbb{Q}}, W), (E_r^{p,q}(K_{\mathcal{O}}, W), F_{\text{rec}}), E_r^{p,q}(\alpha))$$

for all p, q and for $r \geq 1$.

Because F_{rec} on $E_1^{p,q}(K_{\mathcal{O}}, W)$ coincides with the filtration F on $H^{p+q}(\text{Gr}_{-p}^W K_{\mathcal{O}})$ under the isomorphism $E_1^{p,q}(K_{\mathcal{O}}, W) \simeq H^{p+q}(\text{Gr}_{-p}^W K_{\mathcal{O}})$, $E_1^{p,q}(K, W)$ is a pre-variation of \mathbb{Q} -Hodge structure of weight q and the morphism

$$d_1 : E_1^{p,q}(K, W) \rightarrow E_1^{p+1,q}(K, W)$$

is a morphism of pre-variations for all p, q . Thus we conclude that $E_2^{p,q}(K, W)$ is a pre-variation of \mathbb{Q} -Hodge structure of weight q and the morphism d_1 above is strictly compatible with the filtration F for all p, q by Lemma 3.15 (i), (iii). Because the morphism

$$d_2 : E_2^{p,q}(K, W) \longrightarrow E_2^{p+2,q-1}(K, W)$$

preserves the filtration F on both side, we conclude that $d_2 = 0$ for all p, q by Lemma 3.15 (ii). Inductively on r , we obtain $d_r = 0$ for $r \geq 2$, which means the E_2 -degeneracy of the spectral sequence $E_r^{p,q}(K, W)$. Thus we obtain the conclusions (3.20.1) and (3.20.2). Moreover, if K is graded polarizable, then $E_1^{p,q}(K, W)$ is a polarizable pre-variation for all p, q . Then Lemma 3.15 (v) implies that $E_2^{p,q}(K, W)$ is polarizable for all p, q .

The morphism

$$d_0 : E_0^{p,q}(K_{\mathcal{O}}, W) \longrightarrow E_0^{p,q+1}(K_{\mathcal{O}}, W)$$

is strictly compatible with the filtration F_{rec} by (3.16.2). Moreover, the morphism d_r is strictly compatible with the filtration F_{rec} as remarked above. Therefore we obtain the conclusion (3.20.3) by [D3, Proposition (7.2.8)]. \square

Remark 3.21. Let $K = ((K_{\mathbb{Q}}, W), (K_{\mathcal{O}}, W, F), \alpha)$ be a pre-variation of \mathbb{Q} -mixed Hodge complex. If $(K_{\mathcal{O}}, W, F)$ is bifiltered perfect, then $K \otimes^L \mathbb{C}(x)$ is a \mathbb{Q} -mixed Hodge complex for every $x \in X$ and $H^n(K \otimes^L \mathbb{C}(x))$ coincides with $H^n(K) \otimes \mathbb{C}(x)$ as \mathbb{Q} -mixed Hodge structures for every $x \in X$ by Remark 3.18.

Definition 3.22. Let X be a complex manifold.

(i) Let $K = (K_{\mathbb{Q}}, (K_{\mathcal{O}}, F), \alpha)$ be a pre-variation of \mathbb{Q} -Hodge complex of weight m on X . We consider a finite decreasing filtration F on the complex $\Omega_X^1 \otimes K_{\mathcal{O}}$ by

$$F^p(\Omega_X^1 \otimes K_{\mathcal{O}}) = \Omega_X^1 \otimes F^p K_{\mathcal{O}}$$

for all p . A transversal connection on K is a morphism

$$\gamma : (K_{\mathcal{O}}, F) \longrightarrow (\Omega_X^1 \otimes K_{\mathcal{O}}, F[-1])$$

in the filtered derived category of complexes of \mathbb{C} -sheaves such that the diagram

$$\begin{array}{ccc} \mathcal{O}_X \otimes K_{\mathbb{Q}} & \longrightarrow & K_{\mathcal{O}} \\ d \otimes \text{id} \downarrow & & \downarrow \gamma \\ \Omega_X^1 \otimes K_{\mathbb{Q}} & \longrightarrow & \Omega_X^1 \otimes K_{\mathcal{O}} \end{array}$$

commutes in the derived category, where the top and bottom horizontal arrows are the morphisms induced by α .

For a pre-variation of \mathbb{Q} -mixed Hodge complex, a transversal connection is defined by the same way.

(ii) A variation of \mathbb{Q} -(mixed) Hodge complex on X is a pair (K, γ) consisting of a pre-variation of \mathbb{Q} -(mixed) Hodge complex K and a transversal connection γ on K . We simply call K a variation of \mathbb{Q} -(mixed) Hodge complex if there is no danger of confusion.

(iii) Let (K, γ) and (K', γ') be variations of \mathbb{Q} -(mixed) Hodge complexes on X . A morphism of variations of \mathbb{Q} -(mixed) Hodge complexes $\varphi : (K, \gamma) \rightarrow (K', \gamma')$ is a morphism of pre-variations $\varphi : K \rightarrow K'$ which is compatible with the morphism γ and γ' , that is, the diagram

$$\begin{array}{ccc} K_{\mathcal{O}} & \xrightarrow{\varphi_{\mathcal{O}}} & K'_{\mathcal{O}} \\ \gamma \downarrow & & \downarrow \gamma' \\ \Omega_X^1 \otimes K_{\mathcal{O}} & \xrightarrow{\text{id} \otimes \varphi_{\mathcal{O}}} & \Omega_X^1 \otimes K'_{\mathcal{O}} \end{array}$$

is commutative.

(iv) A co-semi-simplicial variation of \mathbb{Q} -(mixed) Hodge complex on X is a covariant functor from the category Δ_{mon} to the category of the variation of \mathbb{Q} -(mixed) Hodge complex on X as usual.

We can easily see the following by the definition above.

Lemma 3.23. *Let $(K^{\bullet}, \gamma^{\bullet})$ be a co-semi-simplicial variation of \mathbb{Q} -mixed Hodge complex on a complex manifold X . Then*

$$(sK^{\bullet}, \delta(W, L), F, s\gamma^{\bullet})$$

is a variation of \mathbb{Q} -mixed Hodge complex on X .

Corollary 3.24. *Let $\varphi : (K_1, \gamma_1) \rightarrow (K_2, \gamma_2)$ be a morphism of variations of \mathbb{Q} -mixed Hodge complex on a complex manifold X . Then the mixed cone $C_M(\varphi)$ with the morphism induced by γ_1, γ_2 is a variation of \mathbb{Q} -mixed Hodge complex.*

Lemma 3.25. *Let X be a complex manifold and (K, γ) a variation of \mathbb{Q} -mixed Hodge complex on X . Then $H^n(K)$ is a variation of \mathbb{Q} -mixed Hodge structure on X for all n .*

In the rest of this section, we consider the case over the unit disc Δ in \mathbb{C} . The following definition and lemmas are technical tools for proving the admissibility of geometric variations of \mathbb{Q} -mixed Hodge structures.

Notation 3.26. We set $\Delta^* = \Delta \setminus \{0\}$. The canonical inclusions $\Delta^* \hookrightarrow \Delta$ and $\{0\} \hookrightarrow \Delta$ are denoted by j and i respectively. The upper

halfplane in \mathbb{C} is denoted by \mathbb{H} and the morphism $\pi : \mathbb{H} \longrightarrow \Delta^*$ is given by $\pi(s) = \exp(2\pi\sqrt{-1}s)$. The nearby cycle functor

$$\psi_t = i^{-1}R(j\pi)_*\pi^{-1} \quad (3.26.1)$$

is a functor from the derived category of the bounded below (filtered) complexes of \mathbb{Q} -sheaves on Δ^* to the derived category of the bounded below (filtered) complexes of \mathbb{Q} -vector spaces.

Lemma 3.27. *Let K be a pre-variation of \mathbb{Q} -Hodge complex on Δ^* . Then we have the canonical isomorphism*

$$H^n\psi_t K_{\mathbb{Q}} \simeq H^0\psi_t H^n(K_{\mathbb{Q}})$$

for all n . Moreover, the canonical morphism

$$R\Gamma(\mathbb{H}, \pi^{-1}K_{\mathbb{Q}}) \longrightarrow \psi_t K_{\mathbb{Q}}$$

is a quasi-isomorphism.

Proof. There exist spectral sequences $E_r^{p,q}(R\Gamma(\mathbb{H}, \pi^{-1}K_{\mathbb{Q}}))$ and $E_r^{p,q}(\psi_t K_{\mathbb{Q}})$ with

$$\begin{aligned} E_2^{p,q}(R\Gamma(\mathbb{H}, \pi^{-1}K_{\mathbb{Q}})) &= H^p(\mathbb{H}, \pi^{-1}H^q(K_{\mathbb{Q}})) \\ E^{p+q}(R\Gamma(\mathbb{H}, \pi^{-1}K_{\mathbb{Q}})) &= H^{p+q}(\mathbb{H}, \pi^{-1}K_{\mathbb{Q}}) \\ E_2^{p,q}(\psi_t K_{\mathbb{Q}}) &= H^p\psi_t H^q(K_{\mathbb{Q}}) = i^{-1}R^p(j\pi)_*\pi^{-1}H^q(K_{\mathbb{Q}}) \\ E^{p+q}(\psi_t K_{\mathbb{Q}}) &= H^{p+q}\psi_t K_{\mathbb{Q}} = i^{-1}R^{p+q}(j\pi)_*\pi^{-1}K_{\mathbb{Q}} \end{aligned}$$

and a morphism

$$E_r^{p,q}(R\Gamma(\mathbb{H}, \pi^{-1}K_{\mathbb{Q}})) \longrightarrow E_r^{p,q}(\psi_t K_{\mathbb{Q}})$$

of spectral sequences.

Because $H^q(K_{\mathbb{Q}})$ is a local system on Δ^* , $\pi^{-1}H^q(K_{\mathbb{Q}})$ is a constant \mathbb{Q} -sheaf on Δ^* . Therefore we have $R^p(j\pi)_*\pi^{-1}H^q(K_{\mathbb{Q}}) = 0$ and $H^p(\mathbb{H}, \pi^{-1}H^q(K_{\mathbb{Q}})) = 0$ for $p \neq 0$. Thus the spectral sequences $E_r^{p,q}(R\Gamma(\mathbb{H}, \pi^{-1}K_{\mathbb{Q}}))$ and $E_r^{p,q}(\psi_t K_{\mathbb{Q}})$ degenerate at E_2 -terms and we have the canonical isomorphisms

$$\begin{aligned} H^n(\mathbb{H}, \pi^{-1}K_{\mathbb{Q}}) &= \Gamma(\mathbb{H}, \pi^{-1}H^n(K_{\mathbb{Q}})) \\ H^n\psi_t K_{\mathbb{Q}} &= H^0\psi_t H^n(K_{\mathbb{Q}}) \end{aligned}$$

for all n . By definition, we have

$$H^0\psi_t H^n(K_{\mathbb{Q}}) = i^{-1}(j\pi)_*\pi^{-1}H^n(K_{\mathbb{Q}})$$

which coincide with $\Gamma(\mathbb{H}, \pi^{-1}H^n(K_{\mathbb{Q}}))$ for all n , because $\pi^{-1}H^n(K_{\mathbb{Q}})$ is a constant \mathbb{Q} -sheaf on \mathbb{H} . \square

Corollary 3.28. *Let K be a pre-variation of \mathbb{Q} -mixed Hodge complex on Δ^* . Then the canonical morphism*

$$(R\Gamma(\mathbb{H}, \pi^{-1}K_{\mathbb{Q}}), W) \longrightarrow \psi_t(K_{\mathbb{Q}}, W)$$

is a filtered quasi-isomorphism. Moreover we have the canonical isomorphism

$$E_r^{p,q}(\psi_t K_{\mathbb{Q}}, W) \simeq \Gamma(\mathbb{H}, \pi^{-1} E_r^{p,q}(K_{\mathbb{Q}}, W))$$

for the spectral sequences. The spectral sequence $E_r^{p,q}(\psi_t K_{\mathbb{Q}}, W)$ degenerates at E_2 -terms.

Proof. Because of the canonical isomorphisms

$$\begin{aligned} \mathrm{Gr}_m^W R\Gamma(\mathbb{H}, \pi^{-1}K_{\mathbb{Q}}) &\simeq R\Gamma(\mathbb{H}, \pi^{-1}\mathrm{Gr}_m^W K_{\mathbb{Q}}) \\ \mathrm{Gr}_m^W \psi_t K_{\mathbb{Q}} &\simeq \psi_t \mathrm{Gr}_m^W K_{\mathbb{Q}} \end{aligned}$$

for all m , the first part is an easy consequence of the lemma above. Therefore the spectral sequence $E_r^{p,q}(\psi_t K_{\mathbb{Q}}, W)$ is isomorphic to the spectral sequence $E_r^{p,q}(R\Gamma(\mathbb{H}, \pi^{-1}K_{\mathbb{Q}}), W)$ whose E_1 -terms are

$$\begin{aligned} E_1^{p,q}(R\Gamma(\mathbb{H}, \pi^{-1}K_{\mathbb{Q}}), W) &= H^{p+q}(\mathbb{H}, \pi^{-1}\mathrm{Gr}_{-p}^W K_{\mathbb{Q}}) \\ &= \Gamma(\mathbb{H}, \pi^{-1}H^{p+q}\mathrm{Gr}_{-p}^W K_{\mathbb{Q}}) \\ &= \Gamma(\mathbb{H}, \pi^{-1}E_1^{p,q}(K_{\mathbb{Q}}, W)) \end{aligned}$$

for all p, q . By using the fact that $E_r^{p,q}(K_{\mathbb{Q}}, W)$ is local system on Δ^* , we can easily see the equality

$$E_r^{p,q}(R\Gamma(\mathbb{H}, \pi^{-1}K_{\mathbb{Q}}), W) = \Gamma(\mathbb{H}, \pi^{-1}E_r^{p,q}(K_{\mathbb{Q}}, W))$$

for all p, q , which implies the E_2 -degeneracy of the spectral sequence $E_r^{p,q}(R\Gamma(\mathbb{H}, \pi^{-1}K_{\mathbb{Q}}), W)$ as expected. \square

Definition 3.29. (i) An extended pre-variation of \mathbb{Q} -mixed Hodge complex on Δ is a data $(K(\log 0), K, A, \zeta, \beta, \eta)$ consisting of

- (3.29.1) a bounded below complex of \mathcal{O}_{Δ} -modules $K(\log 0)$ equipped with a finite increasing filtration W and a finite decreasing filtration F such that $(K(\log 0), W, F)$ is bifiltered perfect,
- (3.29.2) a pre-variation of \mathbb{Q} -mixed Hodge complex K on Δ^* ,
- (3.29.3) a filtered \mathbb{Q} -mixed Hodge complex (for the definition, see [E2, 6.1.4 Definition]) $A = ((A_{\mathbb{Q}}, W^f, W), (A_{\mathbb{C}}, W^f, W, F), \alpha)$,
- (3.29.4) an isomorphism

$$\zeta : (K_{\mathcal{O}}, W, F) \xrightarrow{\simeq} (K(\log 0), W, F)|_{\Delta^*}$$

in the bifiltered derived category,

(3.29.5) a morphism

$$\beta : \psi_t(K_{\mathbb{Q}}, W) \longrightarrow (K_{\mathcal{O}}, W) \otimes^L \mathbb{C}(0)$$

in the filtered derived category,

(3.29.6) a pair $\eta = (\eta_{\mathbb{Q}}, \eta_{\mathbb{C}})$ of an isomorphism

$$\eta_{\mathbb{Q}} : \psi_t(K_{\mathbb{Q}}, W) \longrightarrow (A_{\mathbb{Q}}, W^f)$$

in the filtered derived category and an isomorphism

$$\eta_{\mathbb{C}} : (K(\log 0), W, F) \otimes^L \mathbb{C}(0) \longrightarrow (A_{\mathbb{C}}, W^f, F)$$

in the bifiltered derived category such that the diagram

$$\begin{array}{ccc} \psi_t(K_{\mathbb{Q}}, W) & \xrightarrow{\eta_{\mathbb{Q}}} & (A_{\mathbb{Q}}, W^f) \\ \beta \downarrow & & \downarrow \alpha \\ (K(\log 0), W) \otimes^L \mathbb{C}(0) & \xrightarrow{\eta_{\mathbb{C}}} & (A_{\mathbb{C}}, W^f) \end{array}$$

is commutative in the filtered derived category.

We sometimes use the symbol $K(\log 0)$ or $(K(\log 0), W, F)$ for an extended pre-variation of \mathbb{Q} -mixed Hodge complex. We call K the underlying pre-variation of \mathbb{Q} -Hodge complex and A the limiting filtered \mathbb{Q} -mixed Hodge complex.

(ii) An extended variation of \mathbb{Q} -mixed Hodge complex on Δ is a data $(K(\log 0), \gamma(\log 0), \gamma)$ consisting of

- an extended pre-variation of \mathbb{Q} -mixed Hodge complex $K(\log 0)$ on Δ with the underlying pre-variation K ,
- a morphism $\gamma(\log 0) : (K(\log 0), F) \longrightarrow (\Omega_{\Delta}^1(\log 0) \otimes K(\log 0), F[-1])$ in the filtered derived category
- a morphism

$$\gamma : (K_{\mathcal{O}}, F) \longrightarrow (\Omega_{\Delta^*}^1 \otimes K_{\mathcal{O}}, F[-1])$$

in the filtered derived category

such that

(3.29.7) (K, γ) is a variation of \mathbb{Q} -mixed Hodge complex on Δ^* ,

(3.29.8) the digram

$$\begin{array}{ccc} (K_{\mathcal{O}}, F) & \xrightarrow{\gamma} & (\Omega_{\Delta^*}^1 \otimes K_{\mathcal{O}}, F[-1]) \\ \varsigma \downarrow & & \downarrow \text{id} \otimes \varsigma \\ (K(\log 0), F)|_{\Delta^*} & \xrightarrow{\gamma(\log 0)|_{\Delta^*}} & (\Omega_{\Delta^*}^1 \otimes K(\log 0)|_{\Delta^*}, F[-1]) \end{array}$$

is commutative in the filtered derived category.

We call (K, γ) the underlying variation of \mathbb{Q} -mixed Hodge complex of $(K(\log 0), \gamma(\log 0), \gamma)$. We sometimes omit the symbol γ if there is no danger of confusion.

(iii) For the case of $W_m K(\log 0) = K(\log 0)$, $W_{m-1} K(\log 0) = 0$ for some integer m , we call it an extended (pre-)variation of \mathbb{Q} -Hodge complex of weight m , and omit the filtration W .

(iv) Let

$$\begin{aligned} K(\log 0) &= (K(\log 0), K, A, \zeta, \beta, \eta) \\ K'(\log 0) &= (K'(\log 0), K', A', \zeta', \beta', \eta') \end{aligned}$$

be extended pre-variations of \mathbb{Q} -mixed Hodge complexes on Δ . A morphism of extended pre-variations $K(\log 0) \rightarrow K'(\log 0)$ is a triple $(\sigma(\log 0), \sigma, \bar{\sigma})$ consisting of

- a morphism of bifiltered complexes

$$\sigma(\log 0) : (K(\log 0), W, F) \rightarrow (K'(\log 0), W, F)$$

- a morphism

$$\sigma : K \rightarrow K'$$

of pre-variations of \mathbb{Q} -mixed Hodge complexes on Δ^* ,

- a morphism of filtered \mathbb{Q} -mixed Hodge complex

$$\bar{\sigma} : A \rightarrow A'$$

such that the diagram

$$\begin{array}{ccc} (K_{\mathcal{O}}, W, F) & \xrightarrow{\sigma_{\mathcal{O}}} & (K'_{\mathcal{O}}, W, F) \\ \zeta \downarrow & & \downarrow \zeta' \\ (K(\log 0), W, F)|_{\Delta^*} & \xrightarrow{\sigma(\log 0)|_{\Delta^*}} & (K'(\log 0), W, F)|_{\Delta^*} \end{array}$$

is commutative in the bifiltered derived category and the diagrams

$$\begin{array}{ccc} \psi_t(K_{\mathbb{Q}}, W) & \xrightarrow{\psi_t \sigma_{\mathbb{Q}}} & \psi_t(K'_{\mathbb{Q}}, W) \\ \eta_{\mathbb{Q}} \downarrow & & \downarrow \eta'_{\mathbb{Q}} \\ (A_{\mathbb{Q}}, W^f) & \xrightarrow{\bar{\sigma}_{\mathbb{Q}}} & (A'_{\mathbb{Q}}, W^f) \end{array}$$

$$\begin{array}{ccc} (K(\log 0), W) \otimes^L \mathbb{C}(0) & \xrightarrow{\sigma(\log 0)(0)} & (K'(\log 0), W) \otimes^L \mathbb{C}(0) \\ \eta_{\mathbb{C}} \downarrow & & \downarrow \eta'_{\mathbb{C}} \\ (A_{\mathbb{C}}, W^f) & \xrightarrow{\bar{\sigma}_{\mathbb{C}}} & (A'_{\mathbb{C}}, W^f) \end{array}$$

are commutative in the filtered derived category. We sometimes use the symbol $\sigma(\log 0)$ instead of $(\sigma(\log 0), \sigma, \bar{\sigma})$ for simplicity

(v) Let $(K(\log 0), \gamma(\log 0), \gamma)$ and $(K'(\log 0), \gamma'(\log 0), \gamma')$ be extended variations of \mathbb{Q} -mixed Hodge complexes on Δ . A morphism of extended variations of \mathbb{Q} -mixed Hodge complexes

$$(K(\log 0), \gamma(\log 0), \gamma) \longrightarrow (K'(\log 0), \gamma'(\log 0), \gamma')$$

is a morphism of extended pre-variations of \mathbb{Q} -mixed Hodge complexes $(\sigma(\log 0), \sigma, \bar{\sigma})$ such that $\sigma(\log 0)$ is compatible with $\gamma(\log 0), \gamma'(\log 0)$ and that σ is compatible with γ, γ' in the filtered derived category.

(vi) A co-semi-simplicial extended (pre-)variation of \mathbb{Q} -mixed Hodge complex on Δ is a co-semi-simplicial object in the category of the extended (pre-)variations of \mathbb{Q} -mixed Hodge complexes on Δ . A co-semi-simplicial extended (pre-)variation of \mathbb{Q} -mixed Hodge complex on Δ is said to be *finite* if there exists a non-negative integer q_0 such that $K^q(\log 0) = 0$ for all $q \geq q_0$.

Lemma 3.30. (i) *Let*

$$K^\bullet(\log 0) = (K^\bullet(\log 0), K^\bullet, A^\bullet, \zeta^\bullet, \beta^\bullet, \eta^\bullet)$$

be a finite co-semi-simplicial extended pre-variation of \mathbb{Q} -mixed Hodge complex on Δ . Then

$$(sK^\bullet(\log 0), \delta(W, L), F)$$

is an extended pre-variation of \mathbb{Q} -mixed Hodge complex on Δ whose underlying pre-variation of \mathbb{Q} -mixed Hodge complex is $(sK^\bullet, \delta(W, L), F)$ on Δ^ . For the case where $K^\bullet(\log 0)$ is a finite co-semi-simplicial extended pre-variation of \mathbb{Q} -Hodge complex of weight m ,*

$$(sK^\bullet(\log 0), \delta(L)[m], F)$$

is an extended pre-variation of \mathbb{Q} -mixed Hodge complex whose underlying pre-variation is $(sK^\bullet, \delta(L)[m], F)$.

(ii) *Let*

$$(K^\bullet(\log 0), \gamma^\bullet(\log 0), \gamma^\bullet)$$

be a finite co-semi-simplicial extended variation of \mathbb{Q} -mixed Hodge complex on Δ . Then

$$(sK^\bullet(\log 0), s\gamma^\bullet(\log 0), s\gamma^\bullet)$$

is an extended variation of \mathbb{Q} -mixed Hodge complex on Δ , where the increasing filtration on $sK^\bullet(\log 0)$ is given by $\delta(W, L)$ as in (i) above. For the case where $(K^\bullet(\log 0), \gamma^\bullet(\log 0), \gamma^\bullet)$ is a finite co-semi-simplicial variation of \mathbb{Q} -Hodge complex of weight m ,

$$(sK^\bullet(\log 0), s\gamma^\bullet(\log 0), s\gamma^\bullet)$$

is an extended variation of \mathbb{Q} -mixed Hodge complex on Δ , where the increasing filtration on $sK^\bullet(\log 0)$ is given by $\delta(L)[m]$ as in (i) above.

Proof. We have

$$(\mathrm{Gr}_m^{\delta(W,L)} sK^\bullet(\log 0), F) = \bigoplus_{q \geq 0} (\mathrm{Gr}_{m+q}^W K^q(\log 0)[-q], F) \quad (3.30.1)$$

as filtered complexes, which is, in fact, a finite direct sum because $K^\bullet(\log 0)$ is finite. Thus $(sK^\bullet(\log 0), \delta(W, L), F)$ is bifiltered perfect.

Similarly, we have

$$\begin{aligned} \mathrm{Gr}_m^{\delta(W,L)} \psi_t(sK_\mathbb{Q}^\bullet) &= \mathrm{Gr}_m^{\delta(W,L)} s\psi_t K_\mathbb{Q}^\bullet \\ &= \bigoplus_{q \geq 0} \mathrm{Gr}_{m+q}^W \psi_t K_\mathbb{Q}^q[-q] \end{aligned} \quad (3.30.2)$$

for all m .

By definition, it is clear that $A^\bullet = ((A_\mathbb{Q}^\bullet, W^f, W), (A_\mathbb{C}^\bullet, W^f, W, F), \alpha^\bullet)$ is a co-semi-simplicial filtered \mathbb{Q} -mixed Hodge complex. Therefore

$$\begin{aligned} (sA^\bullet, \delta(W^f, L), \delta(W, L), F) \\ = ((sA_\mathbb{Q}^\bullet, \delta(W^f, L), \delta(W, L)), (sA_\mathbb{C}^\bullet, \delta(W^f, L), \delta(W, L), F), s\alpha^\bullet) \end{aligned}$$

is a filtered \mathbb{Q} -mixed Hodge complex by [E2, 6.1.15 Proposition].

Then we obtain the conclusion by (3.30.1) for $sK^\bullet(\log 0)$ and for sA^\bullet , by (3.30.2) and by Lemma 3.19.

The second part concerning with the variation of \mathbb{Q} -mixed Hodge complex is an easy consequence of the first part and Lemma 3.23. \square

Corollary 3.31. (i) *Let $K(\log 0)$ and $K'(\log 0)$ be two extended pre-variations of \mathbb{Q} -mixed Hodge complex on Δ and*

$$\sigma(\log 0) : K(\log 0) \longrightarrow K'(\log 0)$$

a morphism of extended pre-variations of \mathbb{Q} -mixed Hodge complexes. Then the mixed cone $C_M(\sigma(\log 0))$ is an extended pre-variation of \mathbb{Q} -mixed Hodge complex on Δ .

(ii) *Let $(K(\log 0), \gamma(\log 0), \gamma)$ and $(K'(\log 0), \gamma'(\log 0), \gamma')$ be two extended variations of \mathbb{Q} -mixed Hodge complexes on Δ and*

$$\sigma(\log 0) : (K(\log 0), \gamma(\log 0), \gamma) \longrightarrow (K'(\log 0), \gamma'(\log 0), \gamma')$$

a morphism of extended variations of \mathbb{Q} -mixed Hodge complexes. Then the mixed cone $(C_M(\sigma(\log 0)), C(\gamma(\log 0), \gamma'(\log 0)), C(\gamma, \gamma'))$ is an extended variation of \mathbb{Q} -mixed Hodge complex on Δ , where $C(\gamma(\log 0), \gamma'(\log 0))$ and $C(\gamma, \gamma')$ denote the morphisms induced by $\gamma(\log 0), \gamma'(\log 0)$ and by γ, γ' .

Lemma 3.32. *Let $K(\log 0)$ be an extended pre-variation of \mathbb{Q} -Hodge complex of weight m on Δ . Then we have the following properties:*

(3.32.1) *$H^n(\mathrm{Gr}_F^p K(\log 0))$ is a locally free \mathcal{O}_Δ -module of finite rank for all n, p .*

(3.32.2) *The canonical morphism*

$$H^n(\mathrm{Gr}_F^p K(\log 0)) \otimes \mathbb{C}(t) \longrightarrow H^n(\mathrm{Gr}_F^p K(\log 0) \otimes^L \mathbb{C}(t))$$

is an isomorphism for all n, p and for every $t \in \Delta$.

(3.32.3) *The spectral sequence $E_r^{p,q}(K(\log 0), F)$ degenerate at E_1 -terms, or equivalently, $H^n(\mathrm{Gr}_F^p K(\log 0)) \simeq \mathrm{Gr}_F^p H^n(K(\log 0))$ for all n, p .*

(3.32.4) *The canonical morphism*

$$\mathrm{Gr}_F^p H^n(K(\log 0)) \otimes \mathbb{C}(t) \longrightarrow \mathrm{Gr}_F^p H^n(K(\log 0) \otimes^L \mathbb{C}(t))$$

is an isomorphism for all n, p and for every $t \in \Delta$.

Proof. We have isomorphisms

$$H^n(\psi_t K_{\mathbb{Q}}) \simeq H^0 \psi_t H^n(K_{\mathbb{Q}}) \simeq \Gamma(\mathbb{H}, \pi^{-1} H^n(K_{\mathbb{Q}}))$$

for all n as in the proof of Lemma 3.27. Because β induces an isomorphism $\psi_t K_{\mathbb{Q}} \otimes \mathbb{C} \simeq K(\log 0) \otimes^L \mathbb{C}(0)$ by the condition (3.29.6), we have the equality

$$\dim H^n(K(\log 0) \otimes^L \mathbb{C}(0)) = \mathrm{rank} H^n(K_{\mathbb{Q}})$$

for all n . On the other hand, we have

$$\dim H^n(K(\log 0) \otimes^L \mathbb{C}(t)) = \mathrm{rank} H^n(K_{\mathbb{Q}})$$

for any $t \in \Delta^*$ by the condition (3.29.4). Therefore the function

$$\Delta \ni t \mapsto \dim H^n(K(\log 0) \otimes^L \mathbb{C}(t))$$

is constant for all n . Because the differential

$$d(t) : (K(\log 0) \otimes \mathbb{C}(t))^p \longrightarrow (K(\log 0) \otimes \mathbb{C}(t))^{p+1}$$

is strictly compatible with the filtration $F(K(\log 0) \otimes^L \mathbb{C}(t))$ for all p and for every $t \in \Delta$ by (3.29.4) together with Remark 3.21 and by (3.29.6) for $K(\log 0) = \mathrm{Gr}_m^W K(\log 0)$, we obtain the conclusions (3.32.1)-(3.32.3) by Lemma 3.5 (iv). We can easily obtain (3.32.4) by (3.32.3), by (3.32.2), and by the strict compatibility of the morphism $d(t)$ with the filtration $F(K(\log 0) \otimes^L \mathbb{C}(t))$. \square

Lemma 3.33. *Let $(K(\log 0), W, F)$ be an extended pre-variation of \mathbb{Q} -mixed Hodge complex on Δ . Then we have the following properties:*

(3.33.1) *The spectral sequence $E_r^{p,q}(K(\log 0), W)$ degenerates at E_2 -terms.*

(3.33.2) *$\mathrm{Gr}_F^p \mathrm{Gr}_m^W H^n(K(\log 0))$ is a locally free \mathcal{O}_Δ -module of finite rank for all m, n, p .*

(3.33.3) *The spectral sequence $E_r^{p,q}(K(\log 0), F)$ degenerates at E_1 -terms.*

(3.33.4) *The canonical morphism*

$$(H^n(K(\log 0)), W, F) \otimes \mathbb{C}(t) \longrightarrow (H^n(K(\log 0) \otimes^L \mathbb{C}(t)), W, F)$$

is an isomorphism for all n and for every $t \in \Delta$.

Proof. Because we have an isomorphism

$$(K(\log 0), W, F) \otimes^L \mathbb{C}(t) \simeq (K_{\mathcal{O}}, W, F) \otimes^L \mathbb{C}(t)$$

for $t \in \Delta^*$ in the bifiltered derived category by (3.29.4), the spectral sequence $E_r^{p,q}((K(\log 0), W) \otimes^L \mathbb{C}(t))$ is isomorphic to $E_r^{p,q}((K_{\mathcal{O}}, W) \otimes \mathbb{C}(t))$ for all $t \in \Delta^*$.

As pointed out in Remark 3.21, $(K_{\mathcal{O}}, W, F) \otimes^L \mathbb{C}(t)$ underlies a \mathbb{Q} -mixed Hodge complex and

$$H^n(K_{\mathcal{O}}) \otimes \mathbb{C}(t) \simeq H^n(K_{\mathcal{O}} \otimes^L \mathbb{C}(t))$$

as bifiltered objects for every $t \in \Delta^*$. Therefore the spectral sequence $E_r^{p,q}((K_{\mathcal{O}}, W) \otimes^L \mathbb{C}(t))$ degenerates at E_2 -terms for every $t \in \Delta^*$. Moreover, we have

$$\begin{aligned} E_2^{p,q}((K_{\mathcal{O}}, W) \otimes^L \mathbb{C}(t)) &\simeq \mathrm{Gr}_{-p}^W H^{p+q}(K_{\mathcal{O}} \otimes^L \mathbb{C}(t)) \\ &\simeq \mathrm{Gr}_{-p}^W (H^{p+q}(K_{\mathcal{O}}) \otimes \mathbb{C}(t)) \\ &\simeq \mathrm{Gr}_{-p}^W H^{p+q}(K_{\mathcal{O}}) \otimes \mathbb{C}(t) \\ &\simeq \mathrm{Gr}_{-p}^W H^{p+q}(K_{\mathbb{Q}}) \otimes \mathbb{C}(t) \end{aligned}$$

for every $t \in \Delta^*$. Therefore we have

$$\dim E_2^{p,q}((K(\log 0), W) \otimes^L \mathbb{C}(t)) = \mathrm{rank} \mathrm{Gr}_{-p}^W H^{p+q}(K_{\mathbb{Q}}),$$

which implies that the function

$$\Delta^* \ni t \mapsto \dim E_2^{p,q}((K(\log 0), W) \otimes^L \mathbb{C}(t))$$

is constant for all p, q .

By (3.29.6), the spectral sequence $E_r^{p,q}((K(\log 0), W) \otimes^L \mathbb{C}(0))$ degenerates at E_2 -terms and the equalities

$$\begin{aligned} \dim E_2^{p,q}((K(\log 0), W) \otimes^L \mathbb{C}(0)) \\ &= \dim E_2^{p,q}(\psi_t K_{\mathbb{Q}}, W) \\ &= \dim \Gamma(\mathbb{H}, \pi^{-1} E_2^{p,q}(K_{\mathbb{Q}}, W)) \\ &= \text{rank } \text{Gr}_{-p}^W H^{p+q}(K_{\mathbb{Q}}) \end{aligned}$$

hold by Corollary 3.28. Therefore the function

$$\Delta \ni t \mapsto \dim E_2^{p,q}((K(\log 0), W) \otimes^L \mathbb{C}(t))$$

is constant for all p, q .

We consider $E_r^{p,q}(K(\log 0), W)$ with the filtration F_{rec} . First we will prove

(3.33.5) the morphism

$$d_r : E_r^{p,q}(K(\log 0), W) \longrightarrow E_r^{p+r, q-r+1}(K(\log 0), W)$$

is strictly compatible with the filtration F_{rec} for all p, q and for $r \geq 0$.

We have $(E_0^{p,q}(K(\log 0), W), F_{\text{rec}}) = (\text{Gr}_{-p}^W K^{p+q}(\log 0), F)$ by definition. Because $(\text{Gr}_{-p}^W K(\log 0), F)$ underlies an extended pre-variation of \mathbb{Q} -Hodge complex of weight $-p$, for all p , we obtain (3.33.5) for $r = 0$ by (3.32.3). By (3.32.3) again, we have

$$\begin{aligned} \text{Gr}_{F_{\text{rec}}}^k E_1^{p,q}(K(\log 0), W) &\simeq \text{Gr}_F^k H^{p+q}(\text{Gr}_{-p}^W K(\log 0)) \\ &\simeq H^{p+q}(\text{Gr}_F^k \text{Gr}_{-p}^W K(\log 0)), \end{aligned}$$

which is a locally free \mathcal{O}_{Δ} -module of finite rank for all k, p, q by (3.32.1). Thus $(E_1^{\bullet,q}(K(\log 0), W), F)$ is a filtered perfect complex on Δ for all q , because W is a finite filtration. We have the canonical isomorphism

$$\text{Gr}_{F_{\text{rec}}}^k E_1^{p,q}(K(\log 0), W) \otimes^L \mathbb{C}(t) \simeq \text{Gr}_{F_{\text{rec}}}^k E_1^{p,q}((K(\log 0), W) \otimes^L \mathbb{C}(t))$$

for all k, p, q and for every $t \in \Delta$ by (3.32.4). Then we have

$$\begin{aligned} (E_1^{\bullet,q}(K(\log 0), W), F_{\text{rec}}) \otimes^L \mathbb{C}(t) \\ \simeq (E_1^{\bullet,q}((K(\log 0), W) \otimes^L \mathbb{C}(t)), F_{\text{rec}}) \end{aligned} \tag{3.33.6}$$

as filtered complexes for all q and for every $t \in \Delta$. Therefore we have

$$\begin{aligned} H^p(E_1^{\bullet,q}(K(\log 0), W) \otimes^L \mathbb{C}(t)) &\simeq H^p(E_1^{\bullet,q}((K(\log 0), W) \otimes^L \mathbb{C}(t))) \\ &\simeq E_2^{p,q}((K(\log 0), W) \otimes^L \mathbb{C}(t)) \end{aligned}$$

for all p, q and for every $t \in \Delta$. Then the function

$$\Delta \ni t \mapsto \dim H^p(E_1^{\bullet,q}(K(\log 0), W) \otimes^L \mathbb{C}(t))$$

is constant, as we remarked above.

Taking into account of Lemma 3.5 (iv) for the filtered perfect complex $(E_1^{\bullet,q}(K(\log 0), W), F_{\text{rec}})$, we consider the complex

$$(E_1^{\bullet,q}(K(\log 0), W), F_{\text{rec}}) \otimes^L \mathbb{C}(t),$$

which is isomorphic to

$$(E_1^{\bullet,q}((K(\log 0), W) \otimes^L \mathbb{C}(t)), F_{\text{rec}})$$

as filtered complexes for all $t \in \Delta$ by (3.33.6).

By (3.29.4), $(K(\log 0), W, F) \otimes^L \mathbb{C}(t)$ is isomorphic to $(K_{\mathcal{O}}, W, F) \otimes^L \mathbb{C}(t)$ in the bifiltered derived category for every $t \in \Delta^*$. Therefore the filtered complex $(E_1^{\bullet,q}((K(\log 0), W) \otimes^L \mathbb{C}(t)), F_{\text{rec}})$ is isomorphic to $(E_1^{\bullet,q}((K_{\mathcal{O}}, W) \otimes^L \mathbb{C}(t)), F_{\text{rec}})$ for $t \in \Delta^*$. Because $(K_{\mathcal{O}}, W, F) \otimes^L \mathbb{C}(t)$ underlies a \mathbb{Q} -mixed Hodge complex for every $t \in \Delta^*$ as remarked above, $(E_1^{p,q}((K_{\mathcal{O}}, W) \otimes^L \mathbb{C}(t)), F_{\text{rec}})$ underlies a \mathbb{Q} -Hodge structure of weight q for all p, q and for every $t \in \Delta^*$. Therefore the morphism of E_1 -terms of the spectral sequence $E_r^{p,q}((K_{\mathcal{O}}, W) \otimes^L \mathbb{C}(t))$ is strictly compatible with the filtration F_{rec} for every $t \in \Delta^*$. Thus the morphism

$$d_1(t) : E_1^{p,q}(K(\log 0), W) \otimes^L \mathbb{C}(t) \longrightarrow E_1^{p+1,q}(K(\log 0), W) \otimes^L \mathbb{C}(t)$$

is strictly compatible with the filtration F_{rec} for every $t \in \Delta^*$.

Because $(K(\log 0), W, F) \otimes^L \mathbb{C}(0)$ is isomorphic to $(A_{\mathbb{C}}, W^f, F)$ in the bifiltered derived category by the condition (3.29.6), the spectral sequence $E_r^{p,q}((K(\log 0), W) \otimes^L \mathbb{C}(0))$ is isomorphic to $E_r^{p,q}(A_{\mathbb{C}}, W^f)$. Moreover $(E_1^{p,q}(A_{\mathbb{C}}, W^f), F_{\text{rec}})$ underlies a \mathbb{Q} -mixed Hodge structure and the morphism of E_1 -terms underlies a morphism of \mathbb{Q} -mixed Hodge structures by [E2, 6.1.8 Théorème]. Therefore the morphism of E_1 -terms are strictly compatible with the filtration F_{rec} , which implies that the morphism

$$d_1(0) : E_1^{p,q}(K(\log 0), W) \otimes^L \mathbb{C}(0) \longrightarrow E_1^{p+1,q}(K(\log 0), W) \otimes^L \mathbb{C}(0)$$

is strictly compatible with the filtration F_{rec} by using the isomorphism (3.33.6) for $t = 0$ again.

Thus Lemma 3.5 (iv) assures us that (3.33.5) is valid for $r = 1$, that $H^p(\text{Gr}_{F_{\text{rec}}}^k E_1^{\bullet,q}(K(\log 0), W))$ is locally free, and that the canonical morphism

$$\begin{aligned} H^p(\text{Gr}_{F_{\text{rec}}}^k E_1^{\bullet,q}(K(\log 0), W)) \otimes^L \mathbb{C}(t) \\ \longrightarrow H^p(\text{Gr}_{F_{\text{rec}}}^k (E_1^{\bullet,q}(K(\log 0), W) \otimes^L \mathbb{C}(t))) \end{aligned}$$

is an isomorphism for all k, p, q and for every $t \in \Delta$. Therefore we have the canonical isomorphisms

$$\begin{aligned} \mathrm{Gr}_{F_{\mathrm{rec}}}^k E_2^{p,q}(K(\log 0), W) &\simeq H^p(\mathrm{Gr}_{F_{\mathrm{rec}}}^k E_1^{\bullet,q}(K(\log 0), W)) \\ \mathrm{Gr}_{F_{\mathrm{rec}}}^k E_2^{p,q}(K(\log 0), W) \otimes^L \mathbb{C}(t) &\simeq \mathrm{Gr}_{F_{\mathrm{rec}}}^k E_2^{p,q}((K(\log 0), W) \otimes^L \mathbb{C}(t)) \end{aligned}$$

for all p, q and for every $t \in \Delta$ by the strict compatibility of d_1 and $d_1(t)$ with the filtration F_{rec} . We obtain the canonical identification

$$\begin{aligned} (E_2^{p,q}(K(\log 0), W), F_{\mathrm{rec}}) \otimes^L \mathbb{C}(t) \\ \simeq (E_2^{p,q}((K(\log 0), W) \otimes^L \mathbb{C}(t)), F_{\mathrm{rec}}) \end{aligned}$$

as filtered objects for all p, q . By this identification, we can easily check that the morphism

$$d_r : E_r^{p,q}(K(\log 0), W) \longrightarrow E_r^{p+r, q-r+1}(K(\log 0), W)$$

is the zero morphism for all p, q and for $r \geq 2$, which implies the conclusion (3.33.1). Moreover, we obtain (3.33.5) for all $r \geq 0$.

Then we obtain (3.33.3) by [D3, Proposition (7.2.8)]. Moreover, we have the canonical identification

$$(E_2^{p,q}(K(\log 0), W), F_{\mathrm{rec}}) \simeq (\mathrm{Gr}_{-p}^W H^{p+q}(K(\log 0)), F)$$

for all p, q by using the fact that $E_r^{p,q}(K(\log 0), W)$ degenerates at E_2 -terms. Thus we conclude that

$$\mathrm{Gr}_F^p \mathrm{Gr}_m^W H^n(K(\log 0)) \simeq \mathrm{Gr}_{F_{\mathrm{rec}}}^p E_2^{-m, n+m}(K(\log 0), W)$$

is locally free of finite rank for all m, n, p and that the canonical morphism

$$\mathrm{Gr}_F^p \mathrm{Gr}_m^W H^n(K(\log 0)) \otimes \mathbb{C}(t) \longrightarrow \mathrm{Gr}_F^p \mathrm{Gr}_m^W H^n(K(\log 0) \otimes^L \mathbb{C}(t))$$

is an isomorphism for all m, n, p and for every $t \in \Delta$. Then we obtain (3.33.4). \square

Remark 3.34. Now we remark the notation on the shift of increasing filtrations. We set

$$W[m]_k = W_{k-m}$$

as in [D2], [E2]. Our notation is different from the one in [CKS].

Lemma 3.35. *Let $((A_{\mathbb{Q}}, W^f, W), (A_{\mathbb{C}}, W^f, W, F), \alpha)$ be a filtered \mathbb{Q} -mixed Hodge complex such that the spectral sequence $E_r^{p,q}(A_{\mathbb{C}}, W^f)$ degenerates at E_2 -terms, and $\nu : A_{\mathbb{C}} \longrightarrow A_{\mathbb{C}}$ a morphism of complexes preserving the filtration W^f and satisfying the condition $\nu(W_m A_{\mathbb{C}}) \subset W_{m-2} A_{\mathbb{C}}$ for every m . If the filtration $W[-m]$ on $H^n(\mathrm{Gr}_m^{W^f} A_{\mathbb{C}})$ is the monodromy weight filtration of the endomorphism $H^n(\mathrm{Gr}_m^{W^f} \nu)$ for all m, n , then the filtration W on $H^n(A_{\mathbb{C}})$ is the relative weight monodromy*

filtration of the endomorphism $H^n(\nu)$ with respect to the filtration W^f for all n .

Proof. The assumption implies that the morphism $H^{p+q}(\mathrm{Gr}_{-p}^{W^f} \nu)^k$ induces an isomorphism

$$\mathrm{Gr}_{q+k}^{W[p+q]} E_1^{p,q}(A_{\mathbb{C}}, W^f) \longrightarrow \mathrm{Gr}_{q-k}^{W[p+q]} E_1^{p,q}(A_{\mathbb{C}}, W^f)$$

for all p, q and for $k \geq 0$ because of the isomorphism $E_1^{p,q}(A_{\mathbb{C}}, W^f) \simeq H^{p+q}(\mathrm{Gr}_{-p}^{W^f} A_{\mathbb{C}})$. On the other hand, the E_2 -degeneracy for the filtration W^f gives us the isomorphism

$$E_2^{p,q}(A_{\mathbb{C}}, W^f) \simeq \mathrm{Gr}_{-p}^{W^f} H^{p+q}(A_{\mathbb{C}})$$

for all p, q , under which the filtration W_{rec} on the left hand side coincides with the filtration W on the right hand side by [E2, 6.1.8 Théorème]. Since the morphism d_1 of E_1 -terms induces a morphism of mixed Hodge structures

$$d_1 : (E_1^{p,q}(A_{\mathbb{C}}, W^f), W[p+q], F) \longrightarrow (E_1^{p+1,q}(A_{\mathbb{C}}, W^f), W[p+q+1], F)$$

for all p, q by [E2, 6.1.8 Théorème] again, the morphism $(H^{p+q}(\nu))^k$ induces an isomorphism

$$\mathrm{Gr}_{q+k}^{W[p+q]} \mathrm{Gr}_{-p}^{W^f} H^{p+q}(A_{\mathbb{C}}) \longrightarrow \mathrm{Gr}_{q-k}^{W[p+q]} \mathrm{Gr}_{-p}^{W^f} H^{p+q}(A_{\mathbb{C}})$$

for p, q and for $k \geq 0$. Then we can easily check the conclusion. \square

4. VARIATION OF MIXED HODGE STRUCTURE OF GEOMETRIC ORIGIN

In this section, we discuss variations of mixed Hodge structures arising from mixed Hodge structures on compact support cohomology groups of quasi-projective varieties. We will check that those variations of mixed Hodge structures are *graded polarizable* and *admissible*. These properties will play crucial roles in the subsequent sections.

4.1. Let X and Y be complex manifolds and $f : X \longrightarrow Y$ a smooth morphism. We denote by F the stupid filtration on Ω_X and $\Omega_{X/Y}$. The inclusion $\mathbb{Q}_X \longrightarrow \mathcal{O}_X$ induces a morphism of complexes $\mathbb{Q}_X \longrightarrow \Omega_{X/Y}$. Then we obtain a morphism

$$Rf_* \mathbb{Q}_X \longrightarrow Rf_* \Omega_{X/Y}$$

which is denoted by $\alpha_{X/Y}$.

The differential $d : \Omega_X^p \longrightarrow \Omega_X^{p+1}$ induces a morphism of \mathbb{C} -sheaves

$$d : \Omega_X^p / (f^* \Omega_Y^2 \wedge \Omega_X^{p-2}) \longrightarrow \Omega_X^{p+1} / (f^* \Omega_Y^2 \wedge \Omega_X^{p-1})$$

for all p , where $f^*\Omega_Y^2 \wedge \Omega_X^p$ denotes the image of $f^*\Omega_Y^2 \otimes \Omega_X^p$ in Ω_X^{p+2} by the wedge product of differential forms. Thus we obtain a complex denoted by $\Omega_X/(f^*\Omega_Y^2 \wedge \Omega_X)$ equipped with the finite decreasing filtration F induced by the stupid filtration on Ω_X . We have an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & f^*\Omega_Y^1 \otimes \Omega_{X/Y}[-1] & \longrightarrow & \Omega_X/(f^*\Omega_Y^2 \wedge \Omega_X) & & \\ & & & & \longrightarrow & \Omega_{X/Y} & \longrightarrow 0 \end{array}$$

as in [KO]. We can easily check that the exact sequence above defines a morphism

$$(\Omega_{X/Y}, F) \longrightarrow (f^*\Omega_Y^1 \otimes \Omega_{X/Y}, F[-1])$$

in the filtered derived category of complexes of \mathbb{C} -sheaves. By taking direct images, we obtain a morphism

$$\gamma_{X/Y} : (Rf_*\Omega_{X/Y}, F) \longrightarrow (\Omega_Y^1 \otimes Rf_*\Omega_{X/Y}, F[-1])$$

in the filtered derived category of complexes of \mathbb{C} -sheaves by using the adjunction formula.

The next lemma is a reformulation of the well-known result for our purpose.

Lemma 4.2. *Let X and Y be complex manifolds and $f : X \longrightarrow Y$ a projective smooth morphism. Then*

$$((Rf_*\mathbb{Q}_X, (Rf_*\Omega_{X/Y}, F), \alpha_{X/Y}), \gamma_{X/Y})$$

is a polarizable variation of \mathbb{Q} -Hodge complex of weight 0 on Y .

Notation 4.3. In the situation above, we set

$$K_{X/Y} = (Rf_*\mathbb{Q}_X, (Rf_*\Omega_{X/Y}, F), \alpha_{X/Y}) \quad (4.3.1)$$

which is a polarizable pre-variation of \mathbb{Q} -Hodge complex of weight 0.

Remark 4.4. The construction above is functorial in the following sense. Let

$$\begin{array}{ccc} X_1 & \xrightarrow{g} & X_2 \\ f_1 \downarrow & & \downarrow f_2 \\ Y & \xlongequal{\quad} & Y \end{array}$$

be a commutative diagram of morphisms of complex manifolds such that f_1 and f_2 are projective smooth morphisms. Then the canonical morphisms $Rf_{2*}\mathbb{Q}_{X_2} \longrightarrow Rf_{1*}\mathbb{Q}_{X_1}$ and $Rf_{2*}\Omega_{X_2/Y} \longrightarrow Rf_{1*}\Omega_{X_1/Y}$ induces a morphism $g^* : (K_{X_2/Y}, \gamma_{X_2/Y}) \longrightarrow (K_{X_1/Y}, \gamma_{X_1/Y})$ of variation of \mathbb{Q} -Hodge complexes.

Notation 4.5. For an augmented semi-simplicial variety $f : X^\bullet \longrightarrow Y$, we say f is smooth, projective etc., if $f_q : X^q \longrightarrow Y$ is smooth, projective etc. for all q .

Lemma 4.6. *Let $f : X^\bullet \longrightarrow Y$ be a smooth projective augmented semi-simplicial complex variety. Moreover, we assume that Y is smooth. Then the data*

$$(K_{X^\bullet/Y}, \gamma_{X^\bullet/Y}) = \{(K_{X^q/Y}, \gamma_{X^q/Y})\}_{q \geq 0}$$

is a co-semi-simplicial polarizable variation of \mathbb{Q} -Hodge complex of weight 0. Therefore

$$((sK_{X^\bullet/Y}, \delta(L), F), s\gamma_{X^\bullet/Y})$$

is a graded polarizable variation of \mathbb{Q} -mixed Hodge complex on Y .

Proof. We may change $Rf_{q*}\mathbb{Q}_{X^q}$ (resp. $(Rf_{q*}\Omega_{X^q/Y}, F)$) to the quasi-isomorphic one (resp. filtered quasi-isomorphic one) freely as in Remark 3.17. Therefore we conclude the former half by using Godement resolution (or by other appropriate resolution) and by Lemma 4.2. The latter half is the consequence of Lemma 3.23. \square

Remark 4.7. The construction above is compatible with base change in the following sense. Let $f : X^\bullet \longrightarrow Y$ be as in the lemma above and $g : Z \longrightarrow Y$ a morphism of complex manifolds. Then the fiber product $f_Z : X^\bullet \times_Y Z \longrightarrow Z$ is a smooth projective augmented semi-simplicial complex manifold, and the canonical morphism

$$g^{-1}Rf_*\mathbb{Q}_{X^\bullet} \longrightarrow R(f_Z)_*\mathbb{Q}_{X^\bullet \times_Y Z}$$

underlies a morphism of variation of \mathbb{Q} -mixed Hodge complexes.

Moreover the construction above is functorial in the following sense. Let $f : X^\bullet \longrightarrow Y$ and $f' : X'^\bullet \longrightarrow Y$ be as in the lemma above. Let us assume that a commutative diagram of morphisms

$$\begin{array}{ccc} X'^\bullet & \xrightarrow{g_\bullet} & X^\bullet \\ f' \downarrow & & \downarrow f \\ Y & \xlongequal{\quad} & Y \end{array}$$

are given. Then g_\bullet induces a morphism

$$g_\bullet^* : (K_{X^\bullet/Y}, \gamma_{X^\bullet/Y}) \longrightarrow (K_{X'^\bullet/Y}, \gamma_{X'^\bullet/Y})$$

of co-semi-simplicial variations of \mathbb{Q} -Hodge complexes.

Definition 4.8. Let X, Y be smooth varieties, Z a simple normal crossing divisor on X and $f : X \longrightarrow Y$ a projective morphism. We denote by $j : Z \hookrightarrow X$ the canonical inclusion and by $\varepsilon : Z^\bullet \longrightarrow Z$

the Mayer–Vietoris semi-simplicial resolution of Z . We say that $f : (X, Z) \rightarrow Y$ is smooth if $f : X \rightarrow Y$ and $fj\varepsilon : Z^\bullet \rightarrow Y$ are smooth morphisms.

Lemma 4.9. *Let X, Y, Z and $f : X \rightarrow Y$ be as in Definition 4.8. In addition, we assume $f : (X, Z) \rightarrow Y$ is smooth. We regard $K_{X/Y}$ in (4.3.1) as a variation of \mathbb{Q} -mixed Hodge complex by setting $W_{-1}K_{X/Y} = 0$ and $W_0K_{X/Y} = K_{X/Y}$. Then the morphism $j\varepsilon : Z^\bullet \rightarrow X$ induces a morphism of variations of \mathbb{Q} -mixed Hodge complexes $(j\varepsilon)^* : K_{X/Y} \rightarrow K_{Z^\bullet/Y}$.*

(i) *The mixed cone*

$$K_{(X,Z)/Y}^c = C_M((j\varepsilon)^*)$$

equipped with the morphism

$$\gamma_{(X,Z)/Y}^c : K_{(X,Z)/Y}^c \rightarrow \Omega_Y^1 \otimes K_{(X,Z)/Y}^c$$

induced from the morphisms $\gamma_{X/Y}$ and $s\gamma_{X^\bullet/Y}$ is a graded polarizable variation of \mathbb{Q} -mixed Hodge complex on Y .

(ii) *The canonical morphism*

$$\iota! \mathbb{Q}_{X \setminus Z} \rightarrow \mathbb{Q}_X$$

induces an isomorphism

$$Rf_* \iota! \mathbb{Q}_{X \setminus Z}[1] \simeq (K_{(X,Z)/Y}^c)_{\mathbb{Q}}$$

for the \mathbb{Q} -structure of $K_{(X,Z)/Y}^c$.

(iii) *The canonical morphism*

$$\Omega_{X/Y}(\log Z) \otimes \mathcal{O}_X(-Z) \rightarrow \Omega_{X/Y}$$

induces an isomorphism

$$Rf_* \Omega_{X/Y}^p(\log Z) \otimes \mathcal{O}_X(-Z)[-p+1] \rightarrow \mathrm{Gr}_F^p(K_{(X,Z)/Y}^c)_{\mathcal{O}}$$

for all p .

Proof. We easily obtain the first part by Lemma 3.24.

From the exact sequence

$$0 \longrightarrow \iota! \mathbb{Q}_{X \setminus Z} \longrightarrow \mathbb{Q}_X \longrightarrow j_* \mathbb{Q}_Z \longrightarrow 0$$

we obtain the distinguished triangle

$$Rf_* \iota! \mathbb{Q}_{X \setminus Z} \longrightarrow Rf_* \mathbb{Q}_X \longrightarrow R(fj\varepsilon)_* \mathbb{Q}_{Z^\bullet} \xrightarrow{+1}$$

by using the canonical isomorphism $\mathbb{Q}_Z \rightarrow \varepsilon_* \mathbb{Q}_{Z^\bullet}$. Thus the underlying \mathbb{Q} -structure of $K_{(X,Z)/Y}^c$ is isomorphic to $Rf_* \iota! \mathbb{Q}_{X \setminus Z}[1]$ in the derived category.

We consider the subcomplex $\Omega_{X/Y}(\log Z) \otimes \mathcal{O}_X(-Z)$ of $\Omega_{X/Y}$ equipped with the stupid filtration F . Since the composite of the inclusion

$\Omega_{X/Y}(\log Z) \otimes \mathcal{O}_X(-Z) \hookrightarrow \Omega_{X/Y}$ and the canonical morphism $\Omega_{X/Y} \longrightarrow j_* \varepsilon_* \Omega_{Z^\bullet/Y}$ is the zero morphism, we obtain the morphism

$$Rf_* \Omega_{X/Y}(\log Z) \otimes \mathcal{O}_X(-Z)[1] \longrightarrow (K_{(X,Z)/Y}^c)_{\mathcal{O}} \quad (4.9.1)$$

which preserves the filtration F on the both sides. Because the sequence of the canonical morphism

$$\begin{aligned} 0 \longrightarrow \Omega_{X/Y}^p(\log Z) \otimes \mathcal{O}_X(-Z) &\longrightarrow \Omega_X^p \longrightarrow j_* \varepsilon_0 \Omega_{Z^0}^p \\ &\longrightarrow j_* \varepsilon_1 \Omega_{Z^1}^p \longrightarrow \dots \end{aligned}$$

is exact for all p , we can easily check that the morphism (4.9.1) is isomorphism in the filtered derived category with respect to the filtration F . Therefore we have

$$\begin{aligned} Rf_* \Omega_{X/Y}^p(\log Z) \otimes \mathcal{O}_X(-Z)[-p+1] \\ \simeq Rf_* \mathrm{Gr}_F^p \Omega_{X/Y}(\log Z) \otimes \mathcal{O}_X(-Z)[1] \\ \simeq \mathrm{Gr}_F^p (K_{(X,Z)/Y}^c)_{\mathcal{O}} \end{aligned}$$

for all p . □

Remark 4.10. The construction above is functorial in the following sense. Let X_1, X_2, Y be complex manifolds, Z_1, Z_2 reduced simple normal crossing divisors on X_1, X_2 respectively, and $f_1 : X_1 \longrightarrow Y, f_2 : X_2 \longrightarrow Y$ projective smooth morphisms satisfying the conditions above. Moreover we assume that we have a morphism $h : X_1 \longrightarrow X_2$ with the property $h^{-1}(Z_2) = Z_1$, which commutes with the morphisms f_1 and f_2 . Then we obtain a (non-canonical) morphism of semi-simplicial varieties $h_\bullet : Z_1^\bullet \longrightarrow Z_2^\bullet$ which commutes with the morphism h . Therefore we have commutative diagrams

$$\begin{array}{ccc} Rf_{2*} \mathbb{Q}_{X_2} & \longrightarrow & Rg_{2*} \mathbb{Q}_{Z_2^\bullet} \\ \downarrow & & \downarrow \\ Rf_{1*} \mathbb{Q}_{X_1} & \longrightarrow & Rg_{1*} \mathbb{Q}_{Z_1^\bullet} \end{array} \quad \begin{array}{ccc} Rf_{2*} \Omega_{X_2} & \longrightarrow & Rg_{2*} \Omega_{Z_2^\bullet} \\ \downarrow & & \downarrow \\ Rf_{1*} \Omega_{X_1} & \longrightarrow & Rg_{2*} \Omega_{Z_2^\bullet} \end{array}$$

from which we obtain a morphism

$$h^* : K_{(X_2, Z_2)/Y}^c \longrightarrow K_{(X_1, Z_1)/Y}^c$$

of variations of \mathbb{Q} -mixed Hodge complexes.

Corollary 4.11. *Let $f : X^\bullet \longrightarrow Y$ be a projective augmented semi-simplicial variety and Z^\bullet a semi-simplicial closed subspace of X^\bullet such that Z^q is a simple normal crossing divisor on X^q for all q . Moreover,*

we assume that Y is smooth and that $f : (X^\bullet, Z^\bullet) \longrightarrow Y$ is smooth. By setting

$$\begin{aligned} K_{(X^\bullet, Z^\bullet)/Y}^c &= \{K_{(X^q, Z^q)/Y}^c\}_{q \geq 0} \\ \gamma_{(X^\bullet, Z^\bullet)/Y}^c &= \{\gamma_{(X^q, Z^q)/Y}^c\}_{q \geq 0}, \end{aligned}$$

the data

$$(K_{(X^\bullet, Z^\bullet)/Y}^c, \gamma_{(X^\bullet, Z^\bullet)/Y}^c)$$

is a co-semi-simplicial graded polarizable variation of \mathbb{Q} -mixed Hodge complex on Y . Therefore

$$(sK_{(X^\bullet, Z^\bullet)/Y}^c, s\gamma_{(X^\bullet, Z^\bullet)/Y}^c)$$

is a graded polarizable variation of \mathbb{Q} -mixed Hodge complex on Y .

Example 4.12. Let X be a simple normal crossing variety, D a simple normal crossing divisor on X , and $f : X \longrightarrow Y$ a projective morphism. We denote by $\varepsilon : X^\bullet \longrightarrow X$ the Mayer–Vietoris semi-simplicial resolution of X and by D^\bullet the divisor on X^\bullet defined by $D^q = \varepsilon_q^* D$ on X^q for all q . We assume that $f\varepsilon : (X^\bullet, D^\bullet) \longrightarrow Y$ is smooth. By Corollary 4.11 above, we obtain a variation of \mathbb{Q} -mixed Hodge complex $(sK_{(X^\bullet, D^\bullet)/Y}^c, s\gamma_{(X^\bullet, D^\bullet)/Y}^c)$ on Y , which we denote by $(K_{(X, D)/Y}^c, \gamma_{(X, D)/Y}^c)$.

We have

$$(K_{(X, D)/Y}^c)_{\mathbb{Q}} = s(K_{(X^\bullet, D^\bullet)/Y}^c)_{\mathbb{Q}} \simeq sRf_{\bullet*} \iota_{\bullet!} \mathbb{Q}_{X \setminus D^\bullet}[1] \simeq Rf_{*} \iota_{!} \mathbb{Q}_{X \setminus D}[1]$$

by using the isomorphism $\iota_{!} \mathbb{Q}_{X \setminus D} \simeq \varepsilon_{*} \iota_{\bullet!} \mathbb{Q}_{X^\bullet \setminus D^\bullet}$ in the derived category. Therefore $R^k f_{*} \iota_{!} \mathbb{Q}_{X \setminus D}$ underlies a variation of \mathbb{Q} -mixed Hodge structure on Y for all k . We have

$$\begin{aligned} \mathrm{Gr}_F^p(\mathcal{O}_Y \otimes R^k f_{*} \iota_{!} \mathbb{Q}_{X \setminus D}) &\simeq \mathrm{Gr}_F^p H^{k-1}(K_{(X, D)/Y}^c)_{\mathcal{O}} \\ &\simeq H^{k-1}(\mathrm{Gr}_F^p(K_{(X, D)/Y}^c)_{\mathcal{O}}) \\ &= H^{k-1}(s\mathrm{Gr}_F^p(K_{(X^\bullet, D^\bullet)/Y}^c)_{\mathcal{O}}) \\ &\simeq H^{k-p}(sRf_{\bullet*} \Omega_{X^\bullet/Y}^p(\log D^\bullet) \otimes \mathcal{O}_{X^\bullet}(-D^\bullet)) \\ &= R^{k-p} f_{*} \varepsilon_{*}(\Omega_{X^\bullet/Y}^p(\log D^\bullet) \otimes \mathcal{O}_{X^\bullet}(-D^\bullet)) \end{aligned} \tag{4.12.1}$$

for all k, p . In particular, we have

$$\begin{aligned} \mathrm{Gr}_F^0(\mathcal{O}_Y \otimes R^k f_{*} \iota_{!} \mathbb{Q}_{X \setminus D}) &\simeq R^k f_{*} \varepsilon_{*} \mathcal{O}_{X^\bullet}(-D^\bullet) \\ &\simeq R^k f_{*} \mathcal{O}_X(-D) \end{aligned}$$

for all k , by using the isomorphism $\mathcal{O}_X(-D) \simeq \varepsilon_{*} \mathcal{O}_{X^\bullet}(-D^\bullet)$ in the derived category. Moreover, the isomorphism (4.12.1) is functorial.

4.13. From now on, we treat the case over the unit disc Δ for a while. We fix a coordinate function t of Δ . The morphism of \mathcal{O}_Δ -modules

$$\text{Res}_0 : \Omega_\Delta^1(\log 0) \longrightarrow \mathcal{O}_\Delta$$

which sends dt/t to 1, is a lifting of the usual Poincaré residue morphism. For any morphism $\varphi : \mathcal{F} \longrightarrow \Omega_\Delta^1(\log D) \otimes \mathcal{F}$, where \mathcal{F} is a sheaf, a complex, etc., $\text{Res}_0(\varphi) : \mathcal{F} \longrightarrow \mathcal{F}$ denotes the composite of φ and $\text{Res}_0 \otimes \text{id}$.

For example, for a log connection $\nabla : \mathcal{V} \longrightarrow \Omega_\Delta^1(\log 0) \otimes \mathcal{V}$, the morphism $\text{Res}_0(\nabla)(0)$ coincides with its residue in the usual sense.

4.14. Now we review Steenbrink's results in [St1], [St2] for the later use.

Let X be a smooth complex variety and $f : X \longrightarrow \Delta$ a projective surjective morphism. We set $X_0 = f^{-1}(\Delta^*)$, $D = f^{-1}(0)$ and $f_0 = f|_{X_0}$. We denote the open immersion $X_0 \hookrightarrow X$ by j_X and the closed immersion $D \hookrightarrow X$ by i_X . We assume that D_{red} is a simple normal crossing divisor on X and $f_0 : X_0 \longrightarrow \Delta^*$ is a smooth morphism. The stupid filtration on $\Omega_X(\log D)$ and $\Omega_{X/\Delta}(\log D)$ are denoted by the same letter F as before.

We set

$$K_{X/\Delta}(\log 0) = Rf_*\Omega_{X/\Delta}(\log D)$$

equipped with the filtration F induced by the stupid filtration on $\Omega_{X/\Delta}(\log D)$. The identity map

$$(K_{X_0/\Delta^*}, F) \longrightarrow (K_{X/\Delta}(\log 0), F)|_{\Delta^*}$$

is denoted by ζ .

An exact sequence

$$\begin{aligned} 0 \longrightarrow f^*\Omega_\Delta^1(\log 0) \otimes \Omega_{X/\Delta}(\log D)[-1] &\longrightarrow \Omega_X(\log D) \\ &\longrightarrow \Omega_{X/\Delta}(\log D) \longrightarrow 0 \end{aligned}$$

gives us a morphism

$$(\Omega_{X/\Delta}(\log D), F) \longrightarrow (f^*\Omega_\Delta^1(\log 0) \otimes \Omega_{X/\Delta}(\log D), F[-1])$$

in the filtered derived category, which induces a morphism

$$\gamma_{X/\Delta}(\log 0) : (K_{X/\Delta}(\log 0), F) \longrightarrow (\Omega_\Delta^1(\log 0) \otimes K_{X/\Delta}(\log 0), F[-1])$$

in the filtered derived category. It is clear that $\gamma_{X/\Delta}(\log 0)|_{\Delta^*}$ coincides with γ_{X_0/Δ^*} as in Lemma 4.2. Thus we obtain a log connection

$$H^k(\gamma_{X/\Delta}(\log 0)) : R^k f_*\Omega_{X/\Delta}(\log D) \longrightarrow \Omega_\Delta^1(\log 0) \otimes R^k f_*\Omega_{X/\Delta}(\log D)$$

whose restriction $H^k(\gamma_{X/\Delta}(\log 0))|_{\Delta^*}$ coincides with the connection ∇ given as in Lemma 4.2.

On the other hand, we have the morphism

$$\mathrm{Res}_0(\gamma_{X/\Delta}(\log 0)) : K_{X/\Delta}(\log 0) \longrightarrow K_{X/\Delta}(\log 0)$$

in the derived category. The morphism

$$\begin{aligned} H^k(\mathrm{Res}_0(\gamma_{X/\Delta}(\log 0)))(0) : \\ R^k f_* \Omega_{X/\Delta}(\log D) \otimes \mathbb{C}(0) \longrightarrow R^k f_* \Omega_{X/\Delta}(\log D) \otimes \mathbb{C}(0) \end{aligned}$$

coincides with the residue $\mathrm{Res}_0(H^k(\gamma_{X/\Delta}(\log 0)))(0)$ of the log connection.

Definition 4.15. In the situation above, we say that f is of unipotent monodromy, if $R^k f_{0*} \mathbb{Q}_{X_0}$ is of unipotent monodromy around the origin for all k .

The lemma below is a reformulation of results in [St1, Section 4], [St2, Section 2].

Lemma 4.16. *Let $f : X \longrightarrow \Delta$ be as above. Moreover, we assume that f is of unipotent monodromy.*

(i) *There exist*

- *a \mathbb{Q} -mixed Hodge complex*

$$A_{X/\Delta} = (((A_{X/\Delta})_{\mathbb{Q}}, W), ((A_{X/\Delta})_{\mathbb{C}}, W, F), \alpha),$$

- *a morphism*

$$\beta : \psi_t(K_{X_0/\Delta^*})_{\mathbb{Q}} \longrightarrow K_{X/\Delta}(\log 0) \otimes^L \mathbb{C}(0)$$

in the derived category, and

- *a pair $\eta = (\eta_{\mathbb{Q}}, \eta_{\mathbb{C}})$ of an isomorphism*

$$\eta_{\mathbb{Q}} : \psi_t(K_{X_0/\Delta^*})_{\mathbb{Q}} \longrightarrow (A_{X/\Delta})_{\mathbb{Q}}$$

in the derived category and an isomorphism

$$\eta_{\mathbb{C}} : K_{X/\Delta}(\log 0) \otimes^L \mathbb{C}(0) \longrightarrow ((A_{X/\Delta})_{\mathbb{C}}, F)$$

in the filtered derived category

such that

$$(K_{X/\Delta}(\log 0), K_{X_0/\Delta^*}, A_{X/\Delta}, \zeta, \beta, \eta, \gamma_{X/\Delta}(\log 0))$$

is an extended variation of \mathbb{Q} -Hodge complex of weight 0.

(ii) *There exists a morphism of complexes*

$$\nu : (A_{X/\Delta})_{\mathbb{C}} \longrightarrow (A_{X/\Delta})_{\mathbb{C}}$$

such that the following properties are satisfied:

$$(4.16.1) \quad \nu(W_m(A_{X/\Delta})_{\mathbb{C}}) \subset W_{m-2}(A_{X/\Delta})_{\mathbb{C}} \text{ for all } m.$$

(4.16.2) W on $H^k((A_{X/\Delta})_{\mathbb{C}})$ is the monodromy weight filtration for the nilpotent morphism $H^k(\nu)$ for all k .

(4.16.3) We have a commutative diagram

$$\begin{array}{ccc} K_{X/\Delta}(\log 0) \otimes^L \mathbb{C}(0) & \xrightarrow{\eta_{\mathbb{C}}} & (A_{X/\Delta})_{\mathbb{C}} \\ \text{Res}_0(\gamma_{X/\Delta}(\log 0))(0) \downarrow & & \downarrow \nu \\ K_{X/\Delta}(\log 0) \otimes^L \mathbb{C}(0) & \xrightarrow{\eta_{\mathbb{C}}} & (A_{X/\Delta})_{\mathbb{C}} \end{array}$$

in the derived category.

(iii) All the data described above is functorial with respect to the morphism $f : X \rightarrow \Delta$.

Proof. We set $X_{\infty} = X_0 \times_{\Delta^*} \mathbb{H}$ and denote the projection $X_{\infty} \rightarrow X_0$ by π_X and the projection $X_{\infty} \rightarrow \mathbb{H}$ by f_{∞} . We have the cartesian squares

$$\begin{array}{ccccccc} X_{\infty} & \xrightarrow{\pi_X} & X_0 & \xrightarrow{j_X} & X & \xleftarrow{i_X} & D \\ f_{\infty} \downarrow & & f_0 \downarrow & & \downarrow f & & \downarrow \\ \mathbb{H} & \xrightarrow{\pi} & \Delta^* & \xrightarrow{j} & \Delta & \xleftarrow{i} & \{0\} \end{array}$$

by definition. We set

$$\psi_f = i_X^{-1} R(j_X \pi_X)_* \pi_X^{-1}$$

as in (3.26.1). Because f and f_0 are proper morphism, we have the canonical isomorphism

$$\pi^{-1} Rf_{0*} \mathbb{Q}_{X_0} \rightarrow Rf_{\infty*} \pi_X^{-1} \mathbb{Q}_{X_0} = Rf_{\infty*} \mathbb{Q}_{X_{\infty}},$$

which induces an isomorphism

$$\psi_t(K_{X_0/\Delta^*})_{\mathbb{Q}} \rightarrow i^{-1} Rf_* R(j_X \pi_X)_* \mathbb{Q}_{X_{\infty}} \simeq R\Gamma(D, \psi_f \mathbb{Q}_{X_0}) \quad (4.16.4)$$

in the derived category. In [St1, Section 4], Steenbrink constructed

- a cohomological \mathbb{Q} -mixed Hodge complex

$$A = ((A_{\mathbb{Q}}, W), (A_{\mathbb{C}}, W, F), \alpha)$$

on D ,

- a morphism

$$\beta_X : \psi_f \mathbb{Q}_{X_0} \rightarrow \Omega_{X/\Delta}(\log 0) \otimes \mathcal{O}_D$$

in the derived category,

- a pair $\eta_X = ((\eta_X)_{\mathbb{Q}}, (\eta_X)_{\mathbb{C}})$ consisting of a morphism

$$(\eta_X)_{\mathbb{Q}} : \psi_f \mathbb{Q}_{X_0} \longrightarrow A_{\mathbb{Q}}$$

in the derived category and a morphism

$$(\eta_X)_{\mathbb{C}} : (\Omega_{X/\Delta}(\log D) \otimes \mathcal{O}_D, F) \longrightarrow (A_{\mathbb{C}}, F)$$

of filtered complexes

satisfying the following:

- (4.16.5) The morphism β_X induces an isomorphism

$$\psi_f \mathbb{Q}_{X_0} \otimes \mathbb{C} = \psi_f \mathbb{C}_{X_0} \longrightarrow \Omega_{X/\Delta}(\log D) \otimes \mathcal{O}_D$$

in the derived category.

- (4.16.6) The diagram

$$\begin{array}{ccc} \psi_f \mathbb{Q}_{X_0} & \xrightarrow{\eta_{\mathbb{Q}}} & A_{\mathbb{Q}} \\ \beta_X \downarrow & & \downarrow \alpha \\ \Omega_{X/\Delta}(\log D) \otimes \mathcal{O}_D & \xrightarrow{\eta_{\mathbb{C}}} & A_{\mathbb{C}} \end{array}$$

is commutative in the derived category.

- (4.16.7) The morphism

$$R\Gamma(D, (\eta_X)_{\mathbb{Q}}) : R\Gamma(D, \psi_f \mathbb{Q}_{X_0}) \longrightarrow R\Gamma(D, A_{\mathbb{Q}})$$

is an isomorphism in the derived category.

Then the morphism

$$R\Gamma(D, (\eta_X)_{\mathbb{C}}) : (R\Gamma(D, \Omega_{X/D}(\log D) \otimes \mathcal{O}_D), F) \longrightarrow (R\Gamma(D, A_{\mathbb{C}}), F)$$

in the filtered derived category induces an isomorphism

$$R\Gamma(D, \Omega_{X/D}(\log D) \otimes \mathcal{O}_D) \longrightarrow R\Gamma(D, A_{\mathbb{C}})$$

in the derived category by forgetting the filtration F .

We set

$$A_{X/\Delta} = ((R\Gamma(D, A_{\mathbb{Q}}), W), (R\Gamma(D, A_{\mathbb{C}}), W, F), R\Gamma(D, \alpha)),$$

which is a \mathbb{Q} -mixed Hodge complex. We obtain an isomorphism

$$\eta_{\mathbb{Q}} : \psi_t(K_{X_0/\Delta^*})_{\mathbb{Q}} \longrightarrow (A_{X/\Delta})_{\mathbb{Q}}$$

by composing the isomorphism $\psi_t(K_{X_0/\Delta^*})_{\mathbb{Q}} \longrightarrow R\Gamma(D, \psi_f \mathbb{Q}_{X_0})$ in (4.16.4) and the isomorphism $R\Gamma(D, (\eta_X)_{\mathbb{Q}})$. Similarly we obtain a morphism

$$\eta_{\mathbb{C}} : (K_{X/\Delta}(\log 0) \otimes^L \mathbb{C}(0), F) \longrightarrow ((A_{X/\Delta})_{\mathbb{C}}, F)$$

by composing the canonical morphism

$$\begin{aligned} & (K_{X/\Delta}(\log 0) \otimes^L \mathbb{C}(0), F) \\ &= (Rf_* \Omega_{X/\Delta}(\log D) \otimes^L \mathbb{C}(0), F) \\ &\longrightarrow (R\Gamma(D, \Omega_{X/D}(\log D) \otimes \mathcal{O}_D), F) \end{aligned} \quad (4.16.8)$$

and the morphism $R\Gamma(D, (\eta_X)_{\mathbb{C}})$. Because the canonical morphism (4.16.8) induces an isomorphism in the derived category as proved in [St1, (2.18) Theorem], the morphism $\eta_{\mathbb{C}}$ gives us an isomorphism in the derived category by forgetting the filtration F . Therefore we obtain the morphism

$$\beta : \psi_t(K_{X_0/\Delta^*})_{\mathbb{Q}} \longrightarrow K_{X/\Delta}(\log 0) \otimes^L \mathbb{C}(0)$$

which makes the diagram

$$\begin{array}{ccc} \psi_t(K_{X_0/\Delta^*})_{\mathbb{Q}} & \xrightarrow{\eta_{\mathbb{Q}}} & A_{\mathbb{Q}} \\ \beta \downarrow & & \downarrow R\Gamma(D, \alpha) \\ K_{X/\Delta}(\log 0) \otimes^L \mathbb{C}(0) & \xrightarrow[\eta_{\mathbb{C}}]{} & A_{\mathbb{C}} \end{array}$$

is commutative.

Now (ii) is the consequence of [St1, (5.9) Theorem] (cf. [Sa1, 4.2.5 Remarque], [GN, (5.2) Théorème], [U, (A.1)]).

In order to prove (i), it remains to prove that the morphism $\eta_{\mathbb{C}}$ is an isomorphism in the *filtered* derived category.

Steenbrink proved that

$$H^{p+q}(\mathrm{Gr}_F^p K_{X/\Delta}(\log 0)) = R^q f_* \Omega_{X/\Delta}^p(\log D)$$

is a locally free \mathcal{O}_{Δ} -module of finite rank for all p, q (see [St2, (2.11) Theorem]). Therefore the spectral sequence $E_r^{p,q}(K_{X/\Delta}(\log D), F)$ degenerates at E_1 -terms because the restriction

$$E_r^{p,q}(K_{X/\Delta}(\log 0), F)|_{\Delta^*} = E_r^{p,q}(K_{X_0/\Delta^*}, F)$$

degenerates at E_1 -terms. Thus we have

$$\begin{aligned} \mathrm{Gr}_F^p R^n f_* \Omega_{X/\Delta}(\log D) &= \mathrm{Gr}_F^p H^n(K_{X/\Delta}(\log 0)) \\ &\simeq H^n(\mathrm{Gr}_F^p K_{X/\Delta}(\log 0)) \\ &\simeq R^{n-p} \Omega_{X/\Delta}^p(\log D) \end{aligned}$$

for all n, p , which is a locally free \mathcal{O}_{Δ} -module of finite rank. On the other hand, the canonical morphism

$$H^n(\mathrm{Gr}_F^p K_{X/\Delta}(\log 0)) \otimes \mathbb{C}(t) \longrightarrow H^n(\mathrm{Gr}_F^p(K_{X/\Delta}(\log 0) \otimes^L \mathbb{C}(t)))$$

is an isomorphism for all n, p and for every $t \in \Delta$ by Lemma 3.5 (ii). By using this isomorphism we can easily check that the spectral sequence $E_r^{p,q}((K_{X/\Delta}(\log 0), F) \otimes^L \mathbb{C}(t))$ degenerates at E_1 -terms for every $t \in \Delta$. Thus we have

$$\begin{aligned}
\mathrm{Gr}_F^p(R^n f_* \Omega_{X/\Delta}(\log D) \otimes \mathbb{C}(0)) & \\
&\simeq \mathrm{Gr}_F^p R^n f_* \Omega_{X/\Delta}(\log D) \otimes \mathbb{C}(0) \\
&= \mathrm{Gr}_F^p H^n(K_{X/\Delta}(\log 0)) \otimes \mathbb{C}(0) \\
&\simeq H^n(\mathrm{Gr}_F^p K_{X/\Delta}(\log 0)) \otimes \mathbb{C}(0) \\
&\simeq H^n(\mathrm{Gr}_F^p (K_{X/\Delta}(\log 0) \otimes^L \mathbb{C}(0))) \\
&\simeq \mathrm{Gr}_F^p H^n(K_{X/\Delta}(\log 0) \otimes^L \mathbb{C}(0))
\end{aligned}$$

for all n, p . Therefore it is sufficient to prove that the morphism $\eta_{\mathbb{C}}$ induces an isomorphism

$$\mathrm{Gr}_F^p(R^n f_* \Omega_{X/\Delta}(\log D) \otimes \mathbb{C}(0)) \xrightarrow{\simeq} \mathrm{Gr}_F^p H^n(D, A_{\mathbb{C}}) \quad (4.16.9)$$

for all n, p . Since $\mathrm{Gr}_F^p R^n f_* \Omega_{X/\Delta}(\log D)$ is locally free for all n, p , Corollary 6.2 below implies that the filtration F on $R^n f_* \Omega_{X/\Delta}(\log D) \otimes \mathbb{C}(0)$ coincides with the filtration obtained by Schmid's nilpotent orbit theorem [Sc, (4.9)]. Then the data

$$((H^n \psi_t(K_{X_0/\Delta^*})_{\mathbb{Q}}, W), (R^n f_* \Omega_{X/\Delta}(\log D) \otimes \mathbb{C}(0), W, F), \beta)$$

is a \mathbb{Q} -mixed Hodge structure by [Sc, (6.16) Theorem], where W is the monodromy weight filtration of the logarithm of the monodromy automorphism. Because the filtration W on $H^n(D, A_{\mathbb{C}})$ coincides with the monodromy weight filtration as we already mentioned in (ii), the morphism

$$(R^n f_* \Omega_{X/\Delta}(\log D) \otimes \mathbb{C}(0), F) \longrightarrow (H^n(D, A_{\mathbb{C}}), F)$$

underlies a morphism of \mathbb{Q} -mixed Hodge structures. Thus we obtain the isomorphism (4.16.9).

Now the functoriality (iii) is clear from the construction in [St1]. \square

Lemma 4.17. *Let $f : X^\bullet \longrightarrow \Delta$ be a smooth projective augmented semi-simplicial complex variety. Moreover, we assume the following:*

(4.17.1) *There exists a positive integer q_0 such that $X^q = \emptyset$ for all $q \geq q_0$.*

(4.17.2) *$f_q : X^q \longrightarrow \Delta$ is smooth over Δ^* .*

(4.17.3) *The divisor $f_q^{-1}(0)_{\mathrm{red}}$ is a simple normal crossing divisor on X^q .*

(4.17.4) f is of unipotent monodromy, that is, $f_q : X^q \rightarrow \Delta$ is of unipotent monodromy for all q .

(i) The data

$$(K_{X^\bullet/\Delta}(\log 0), \gamma_{X^\bullet/\Delta}(\log 0)) = \{(K_{X^q/\Delta}(\log 0), \gamma_{X^q/\Delta}(\log 0))\}_{q \geq 0}$$

is a finite co-semi-simplicial extended variation of \mathbb{Q} -Hodge complex of weight 0, whose underlying co-semi-simplicial \mathbb{Q} -Hodge complex of weight 0 is $(K_{X_0^\bullet/\Delta^*}, \gamma_{X_0^\bullet/\Delta^*})$ given in Lemma 4.6.

(ii) The data

$$((sK_{X^\bullet/\Delta}(\log 0), \delta(L), F), s\gamma_{X^\bullet/\Delta}(\log 0))$$

is an extended variation of \mathbb{Q} -mixed Hodge complex whose underlying variation of \mathbb{Q} -mixed Hodge complex is $(sK_{X_0^\bullet/\Delta^*}, s\gamma_{X_0^\bullet/\Delta^*})$ given in Lemma 4.6.

(iii) For the limiting filtered \mathbb{Q} -mixed Hodge complex

$$A = ((A_{\mathbb{Q}}, W^f, W), (A_{\mathbb{C}}, W^f, W, F), \alpha)$$

of $(sK_{X^\bullet/\Delta}, s\gamma_{X^\bullet/\Delta})$, there exists a morphism of complexes

$$\nu : A_{\mathbb{C}} \rightarrow A_{\mathbb{C}}$$

satisfying the following properties:

(4.17.5) $\nu(W_m^f A_{\mathbb{C}}) \subset W_m^f A_{\mathbb{C}}$ for all m .

(4.17.6) $\nu(W_m A_{\mathbb{C}}) \subset W_{m-2} A_{\mathbb{C}}$ for all m .

(4.17.7) The diagram

$$\begin{array}{ccc} sK_{X^\bullet/\Delta}(\log 0) \otimes \mathbb{C}(0) & \xrightarrow{\eta_{\mathbb{C}}} & A_{\mathbb{C}} \\ \text{Res}_0(s\gamma_{X^\bullet/\Delta}(\log 0))(0) \downarrow & & \downarrow \nu \\ sK_{X^\bullet/\Delta}(\log 0) \otimes \mathbb{C}(0) & \xrightarrow{\eta_{\mathbb{C}}} & A_{\mathbb{C}} \end{array}$$

commutes in the derived category.

(4.17.8) The filtration $W[-m]$ on $H^k(\text{Gr}_m^{W^f} A_{\mathbb{C}})$ coincides with the monodromy weight filtration of the nilpotent endomorphism $H^k(\text{Gr}_m^{W^f} \nu)$ for all k, m .

(4.17.9) The filtration W on $H^k(A_{\mathbb{C}})$ coincides with the monodromy weight filtration of the nilpotent endomorphism $H^k(\nu)$ relative to the filtration W^f for all k .

(iv) These data are functorial with respect to $f : X^\bullet \rightarrow \Delta$.

Proof. We obtain (i) by Lemma 4.16 (i) (iii). Here we note that the assumption (4.17.1) implies that $K_{X^\bullet/\Delta}(\log 0)$ is finite. The second part (ii) is the consequence of Lemma 3.30 (ii).

We proceed to prove (iii). Let $A_{X^\bullet/\Delta} = \{A_{X^q/\Delta}\}_{q \geq 0}$ be the limiting co-semi-simplicial \mathbb{Q} -mixed Hodge complex of $K_{X^\bullet/\Delta}(\log 0)$. Then the limiting filtered \mathbb{Q} -mixed Hodge complex A is given by

$$A = (sA_{X^\bullet/\Delta}, \delta(L), \delta(W, L), F)$$

as in Lemma 3.30. For each q , we have morphisms of complexes

$$\nu_q : (A_{X^q/\Delta})_{\mathbb{C}} \longrightarrow (A_{X^q/\Delta})_{\mathbb{C}}$$

satisfying the conditions (4.16.1)-(4.16.3), which form a morphism of co-semi-simplicial complex

$$\nu_\bullet : (A_{X^\bullet/\Delta})_{\mathbb{C}} \longrightarrow (A_{X^\bullet/\Delta})_{\mathbb{C}} \quad (4.17.10)$$

by the functoriality (iii) in Lemma 4.16.

Setting

$$\nu = s\nu_\bullet : s(A_{X^\bullet/\Delta})_{\mathbb{C}} = A_{\mathbb{C}} \longrightarrow A_{\mathbb{C}} = s(A_{X^\bullet/\Delta})_{\mathbb{C}}$$

it is sufficient to prove that the morphism ν satisfies the properties (4.17.5)-(4.17.9). The properties $\nu(W_m^f A_{\mathbb{C}}) \subset W_m^f A_{\mathbb{C}}$ and $\nu(W_m A_{\mathbb{C}}) \subset W_{m-2} A_{\mathbb{C}}$ are trivial by definition. The commutativity in (4.17.7) is easy to see from $\text{Res}_0(s\gamma_{X^\bullet/\Delta}(\log 0)) = s \text{Res}_0(\gamma_{X^\bullet/\Delta}(\log 0))$.

Since the morphism

$$\text{Gr}_m^{W^f} \nu : \text{Gr}_m^{W^f} A_{\mathbb{C}} \longrightarrow \text{Gr}_m^{W^f} A_{\mathbb{C}}$$

coincides with the morphism $\nu_{-m}[m]$ under the identity

$$\begin{aligned} (\text{Gr}_m^{W^f} A_{\mathbb{C}}, W) &= (\text{Gr}_m^{\delta(L)} s(A_{X^\bullet/\Delta})_{\mathbb{C}}, \delta(W, L)) \\ &= ((A_{X^{-m}/\Delta})_{\mathbb{C}}[m], W[m]) \end{aligned}$$

for all m . Thus we obtain (4.17.8). Then (4.17.9) is a consequence of (4.17.8) by Lemma 3.35.

The functoriality (iv) is more or less trivial. \square

Lemma 4.18. *Let X be a smooth complex variety, Z a reduced simple normal crossing divisor on X and $f : X \longrightarrow \Delta$ a projective morphism. We set $D = f^{-1}(0)$, $j : Z \hookrightarrow X$ and $\varepsilon : Z^\bullet \longrightarrow Z$ as before. We assume the following:*

(4.18.1) $D_{\text{red}} + Z$ is a reduced simple normal crossing divisor on X .

(4.18.2) $f : (X, Z) \longrightarrow \Delta$ is smooth over Δ^* .

(4.18.3) $f : (X, Z) \longrightarrow \Delta$ is of unipotent monodromy, that is, $f : X \longrightarrow \Delta$ and $fj\varepsilon : Z^\bullet \longrightarrow \Delta$ are of unipotent monodromy.

Then the morphism $j\varepsilon : Z^\bullet \longrightarrow X$ induces a morphism

$$(j\varepsilon)^* : K_{X/\Delta}(\log 0) \longrightarrow K_{Z^\bullet/\Delta}(\log 0)$$

of extended variation of \mathbb{Q} -mixed Hodge complexes.

(i) The mixed cone

$$K_{(X,Z)/\Delta}^c(\log 0) = C_M((j\varepsilon)^*)$$

equipped with the morphism $\gamma_{(X,Z)/\Delta}^c(\log 0)$ induced by $\gamma_{X/\Delta}(\log 0)$ and $\gamma_{Z^\bullet/\Delta}$ is an extended variation of \mathbb{Q} -mixed Hodge complex on Δ .

(ii) We denote by

$$A = ((A_{\mathbb{Q}}, W^f, W), (A_{\mathbb{C}}, W^f, W, F), \alpha)$$

the limiting filtered \mathbb{Q} -mixed Hodge complex of $K_{(X,Z)/\Delta}^c(\log 0)$. Then there exists a nilpotent morphism of complexes $\nu : A_{\mathbb{C}} \longrightarrow A_{\mathbb{C}}$ compatible with the filtration W^f , such that

(4.18.4) the filtration $W[-m]$ on $H^k(\mathrm{Gr}_m^{W^f} A_{\mathbb{C}})$ coincides with the monodromy weight filtration of the nilpotent endomorphism $H^k(\mathrm{Gr}_m^{W^f} \nu)$ for all k, m , and

(4.18.5) the filtration W on $H^k(A_{\mathbb{C}})$ coincides with the monodromy weight filtration of the nilpotent endomorphism $H^k(\nu)$ relative to the filtration W^f .

Proof. The first part is a consequence of Corollary 3.31.

We proceed to prove the second part. We denote by $A_{X/\Delta}$ the limiting \mathbb{Q} -mixed Hodge complex of $K_{X/\Delta}(\log 0)$ and by $A_{Z^\bullet/\Delta}$ the limiting co-semi-simplicial filtered \mathbb{Q} -mixed Hodge complex of $K_{Z^\bullet/\Delta}(\log 0)$. We regard $A_{X/\Delta}$ as filtered \mathbb{Q} -mixed Hodge complex by setting $W_{-1}^f A_{X/\Delta} = 0$ and $W_0^f A_{X/\Delta} = A_{X/\Delta}$ as usual. Then the limiting filtered \mathbb{Q} -mixed Hodge complex A is given by the mixed cone $A = C_M(A_{X/\Delta} \longrightarrow sA_{Z^\bullet/\Delta})$ by Corollary 3.31. Therefore we have

$$(\mathrm{Gr}_m^{W^f} A_{\mathbb{C}}, W) = (\mathrm{Gr}_{m-1}^{W^f}(A_{X/\Delta})[1], W[1]) \oplus (\mathrm{Gr}_m^{W^f} sA_{Z^\bullet/\Delta}, W),$$

which implies the equality

$$\begin{aligned} & (H^k(\mathrm{Gr}_m^{W^f} A_{\mathbb{C}}), W) \\ &= (H^{k+1}(\mathrm{Gr}_{m-1}^{W^f}(A_{X/\Delta})_{\mathbb{C}}), W[1]) \oplus (H^k(\mathrm{Gr}_m^{W^f}(sA_{Z^\bullet/\Delta})_{\mathbb{C}}), W) \end{aligned}$$

for all k, m . Thus we can easily obtain (4.18.4), which implies (4.18.5) by Lemma 3.35. \square

Lemma 4.19. *Let $f : X^\bullet \rightarrow \Delta$ be a projective augmented semi-simplicial complex variety and Z^\bullet a semi-simplicial closed subspace of X^\bullet such that Z^q is a simple normal crossing divisor on X^q for all q . We assume the following:*

(4.19.1) *There exists a positive integer q_0 such that $X^q = \emptyset$ for all $q \geq q_0$.*

(4.19.2) *$f_q : (X^q, Z^q) \rightarrow \Delta$ is smooth over Δ^* for all q .*

(4.19.3) *$f_q^{-1}(0)_{\text{red}} + Z^q$ is a reduced simple normal crossing divisor on X^q for all q .*

(4.19.4) *$f : (X^\bullet, Z^\bullet) \rightarrow \Delta$ is unipotent monodromy, that is, $f_q : (X^q, Z^q) \rightarrow \Delta$ is of unipotent monodromy for all q .*

(i) *The data*

$$K_{(X^\bullet, Z^\bullet)/\Delta}^c(\log 0) = \{K_{(X^q, Z^q)/\Delta}^c(\log 0)\}_{q \geq 0}$$

equipped with

$$\gamma_{(X^\bullet, Z^\bullet)/\Delta}^c(\log 0) = \{\gamma_{(X^q, Z^q)/\Delta}^c(\log 0)\}_{q \geq 0}$$

is a co-semi-simplicial extended variation of \mathbb{Q} -mixed Hodge complex on Δ . Therefore

$$(sK_{(X^\bullet, Z^\bullet)/\Delta}^c(\log 0), s\gamma_{(X^\bullet, Z^\bullet)/\Delta}^c(\log 0))$$

is an extended variation of \mathbb{Q} -mixed Hodge complex on Δ .

(ii) *We denote by $A = ((A_{\mathbb{Q}}, W^f, W), (A_{\mathbb{C}}, W^f, W, F), \alpha)$ the limiting filtered \mathbb{Q} -mixed Hodge complex of $sK_{(X^\bullet, Z^\bullet)/\Delta}^c(\log 0)$. Then there exists a nilpotent endomorphism of complexes*

$$\nu : A_{\mathbb{C}} \rightarrow A_{\mathbb{C}}$$

compatible with the filtration W^f , such that the filtration W on $H^k(A_{\mathbb{C}})$ coincides with the monodromy weight filtration of the nilpotent endomorphism $H^k(\nu)$ relative to the filtration W^f .

Proof. The first part is the consequences of Lemma 3.30 (ii).

We denote by

$$A^\bullet = ((A_{\mathbb{Q}}^\bullet, W^f, W), (A_{\mathbb{C}}^\bullet, W^f, W, F), \alpha)$$

the limiting co-semi-simplicial filtered \mathbb{Q} -mixed Hodge complex of $K_{(X^\bullet, Z^\bullet)/Y}^c(\log 0)$. Then we have

$$(\text{Gr}_m^{W^f} A_{\mathbb{C}}, W) = \bigoplus_{q \geq 0} (\text{Gr}_{m+q}^{W^f} A_{\mathbb{C}}^q[q], W[-q]),$$

for all m , which implies

$$(H^k(\mathrm{Gr}_m^{W^f} A_{\mathbb{C}}), W) = \bigoplus_{q \geq 0} (H^{k+q}(\mathrm{Gr}_{m+q}^{W^f} A_{\mathbb{C}}^q), W[-q])$$

for all k, m . Then (4.18.4) and Lemma 3.35 implies the second part as before. \square

The following theorem is the main result of this section. Although it seems to follow from the theory of mixed Hodge modules by Morihiko Saito [Sa2], we give here a detailed proof, because we need an explicit description of the Hodge filtration in Section 5.

Theorem 4.20 (GPVMHS arising from mixed Hodge structures on compact support cohomology groups). *Let X be a complex variety, $Z \subset X$ a closed subvariety of X and $f : X \rightarrow Y$ a projective surjective morphism to a smooth complex variety Y . The open immersion $X \setminus Z \hookrightarrow X$ is denoted by ι . Then there exists a Zariski open dense subset Y_0 of Y such that $(R^n f_* \iota_* \mathbb{Q}_{X \setminus Z})|_{Y_0}$ underlies an admissible graded polarizable variation of \mathbb{Q} -mixed Hodge structure for every n .*

Proof. Without loss of generality, we may assume that all irreducible components of X and Z are dominant onto Y .

We can take a cubical hyperresolution $a : X^\bullet \rightarrow X$ such that X^q is smooth for all q and that $Z^\bullet = a^{-1}(Z)$ is a simple normal crossing divisor on X^\bullet (see e.g. [GNPP, (IV.1.26) Théorème]).

Here we remark that X^\bullet has finitely many irreducible components. We denote the semi-simplicial varieties associated to the cubical varieties X^\bullet by X^\bullet too by abuse of the language. Then we obtain a smooth projective augmented semi-simplicial variety $a : X^\bullet \rightarrow X$ satisfying the following conditions:

(4.20.1) there exists a positive integer q_0 such that $X^q = \emptyset$ for $q \geq q_0$.

(4.20.2) X^q is smooth for all q .

(4.20.3) The morphisms $a : X^\bullet \rightarrow X$ is of cohomological descent.

(4.20.4) There exists a non-empty Zariski open subset Y_0 of Y such that the morphism

$$f \cdot a : (X^\bullet, Z^\bullet) \rightarrow Y$$

is smooth over Y_0 .

We set $f_q = f \cdot a_q : X^q \rightarrow Y$, $X_0 = f^{-1}(Y_0)$ and $X_0^\bullet = a^{-1}(X_0)$ and denote the induced morphisms $X_0 \rightarrow Y_0$ by the same letter f etc. by

abuse of the language. Here we replace the condition (4.20.3) by the weaker condition that

(4.20.5) the morphisms $a : X_0^\bullet \longrightarrow X_0$ is of cohomological descent,

for the later use.

First, we prove that $(R^k f_* \iota_! \mathbb{Q}_{X \setminus Z})|_{Y_0} = R^k f_* \iota_! \mathbb{Q}_{X_0 \setminus Z_0}$ underlies a graded polarizable variation of \mathbb{Q} -mixed Hodge structure for every k . Because $a : X_0^\bullet \longrightarrow X_0$ is of cohomological descent, we have the canonical isomorphism

$$\iota_! \mathbb{Q}_{X_0 \setminus Z_0} \xrightarrow{\simeq} Ra_* a^{-1} \iota_! \mathbb{Q}_{X_0 \setminus Z_0} = Ra_* \iota_{\bullet!} \mathbb{Q}_{X_0^\bullet \setminus Z_0^\bullet}$$

by using the fact $a^{-1}(Z) = Z^\bullet$. Therefore we have

$$Rf_* \iota_! \mathbb{Q}_{X_0 \setminus Z_0} \simeq R(f \cdot a)_* \iota_{\bullet!} \mathbb{Q}_{X_0^\bullet \setminus Z_0^\bullet} = sRf_{\bullet*} \iota_{\bullet!} \mathbb{Q}_{X_0^\bullet \setminus Z_0^\bullet}$$

by definition. Since $Rf_{\bullet*} \iota_{\bullet!} \mathbb{Q}_{X_0^\bullet \setminus Z_0^\bullet}$ is isomorphic to $(K_{(X_0^\bullet, Z_0^\bullet)/Y_0}^c)_{\mathbb{Q}}[-1]$ in the derived category, we have the isomorphism

$$Rf_* \iota_! \mathbb{Q}_{X_0 \setminus Z_0} \simeq (sK_{(X_0^\bullet, Z_0^\bullet)/Y_0}^c)_{\mathbb{Q}}[-1]$$

in the derived category. Therefore $R^k f_* \iota_! \mathbb{Q}_{X_0 \setminus Z_0}$ underlies a graded polarizable variation of \mathbb{Q} -mixed Hodge structure for all k by Corollary 4.11.

Now we prove the admissibility. Because the graded polarizable variation of \mathbb{Q} -mixed Hodge structure constructed above commutes with the base change as in Remark 4.7, we may assume $Y = \Delta$ and $Y_0 = \Delta^*$.

Our variation of graded polarizable \mathbb{Q} -mixed Hodge structure has a \mathbb{Z} -structure. Therefore, the quasi-unipotency of the monodromy around the origin is obvious by Remark 3.14. Thus we have the property (3.12.1) in Definition 3.12 (i).

Once we know the quasi-unipotency of the monodromy, Lemma 1.9.1 in [Ks] allows us to assume that the monodromy automorphism of our graded polarizable variation of \mathbb{Q} -mixed Hodge structure is unipotent and that $f : (X^\bullet, Z^\bullet) \longrightarrow \Delta$ is of unipotent monodromy. Moreover we may assume that $f^{-1}(0)_{\text{red}} + Z^\bullet$ is a simple normal crossing divisor on smooth semi-simplicial complex variety X^\bullet . Here we remark that the condition (4.20.3) were destroyed but the condition (4.20.5) is still valid after these procedures.

Now we can apply Lemma 4.19 and complete the proof. \square

Remark 4.21. The graded polarizable variation of \mathbb{Q} -mixed Hodge structure constructed in the theorem above is independent of the choice of the cubical hyperresolutions. We can prove the independence by the usual way (see e.g. [GNPP, I 3.]).

5. HIGHER DIRECT IMAGES OF LOG CANONICAL DIVISORS

This section is the main part of this paper. The following theorem is our main theorem (cf. [Kw1, Theorem 5], [Ko2, Theorem 2.6], [N1, Theorem 1], [F4, Theorems 3.4 and 3.9], and [Kw3, Theorem 1.1]), which is a *natural* generalization of the Fujita–Kawamata semi-positivity theorem for simple normal crossing pairs. For the notion of *upper* and *lower* canonical extensions, see [Ko2, Section 2].

Theorem 5.1. *Let (X, D) be a simple normal crossing pair such that D is reduced and let $f : X \rightarrow Y$ be a projective surjective morphism onto a smooth algebraic variety Y . Assume that every stratum of (X, D) is dominant onto Y . Let Σ be a simple normal crossing divisor on Y such that every stratum of (X, D) is smooth over $Y_0 = Y \setminus \Sigma$. We put $X_0 = f^{-1}(Y_0)$, $D_0 = D|_{X_0}$, $f_0 = f|_{X_0}$, and $d = \dim X - \dim Y$. Let $\iota : X_0 \setminus D_0 \rightarrow X_0$ be the natural open immersion. Then we obtain*

- (1) $R^k f_{0*} \iota_! \mathbb{Q}_{X_0 \setminus D_0} \simeq R^k (f|_{X_0 \setminus D_0})_! \mathbb{Q}_{X_0 \setminus D_0}$ underlies a graded polarizable variation of \mathbb{Q} -mixed Hodge structure on Y_0 for every k . Moreover, it is admissible.

We put $\mathcal{H}_{Y_0}^k = R^k f_{0*} \iota_! \mathbb{Q}_{X_0 \setminus D_0} \otimes \mathcal{O}_{Y_0}$ for every k . Let

$$\dots \subset \mathcal{F}^{p+1}(\mathcal{H}_{Y_0}^k) \subset \mathcal{F}^p(\mathcal{H}_{Y_0}^k) \subset \mathcal{F}^{p-1}(\mathcal{H}_{Y_0}^k) \subset \dots$$

be the Hodge filtration. We assume that all the local monodromies on $R^{d-i} f_{0*} \iota_! \mathbb{Q}_{X_0 \setminus D_0}$ around Σ are unipotent. Then we obtain

- (2) $R^{d-i} f_* \mathcal{O}_X(-D)$ is isomorphic to the canonical extension of

$$\mathrm{Gr}_F^0(\mathcal{H}_{Y_0}^{d-i}) = \mathcal{F}^0(\mathcal{H}_{Y_0}^{d-i}) / \mathcal{F}^1(\mathcal{H}_{Y_0}^{d-i}).$$

We denote it by $\mathrm{Gr}_F^0(\mathcal{H}_Y^{d-i})$. In particular, $R^{d-i} f_* \mathcal{O}_X(-D)$ is locally free.

By the Grothendieck duality, we obtain

- (3) $R^i f_* \omega_{X/Y}(D)$ is isomorphic to the canonical extension of

$$\mathrm{Gr}_F^0(\mathcal{H}_{Y_0}^{d-i})^* = \mathrm{Hom}_{\mathcal{O}_{Y_0}}(\mathrm{Gr}_F^0(\mathcal{H}_{Y_0}^{d-i}), \mathcal{O}_{Y_0}).$$

Thus, $R^i f_* \omega_{X/Y}(D) \simeq \mathrm{Gr}_F^0(\mathcal{H}_Y^{d-i})^*$. In particular, $R^i f_* \omega_{X/Y}(D)$ is locally free.

- (4) We further assume that Y is complete. Then $R^i f_* \omega_{X/Y}(D)$ is semi-positive. For details of semi-positive locally free sheaves (cf. Definition 6.21), see Section 6 below.

Even the following very special case of Theorem 5.1 has never been checked before. We note that it does not follow from [Kw3, Theorem 1.1] (see 5.6 below).

Corollary 5.2. *Let $f : X \rightarrow Y$ be a projective morphism from a simple normal crossing variety X to a smooth complete algebraic variety Y . Assume that every stratum of X is smooth over Y . Then $R^i f_* \omega_{X/Y}$ is a semi-positive locally free sheaf for every i .*

It is natural to prove Theorem 5.3, which is a slight generalization of (2) and (3) in Theorem 5.1, simultaneously with Theorem 5.1.

Theorem 5.3 (cf. [Ko2, Theorem 2.6]). *We use the same notation and assumptions as in Theorem 5.1. We do not assume that the local monodromies on $R^{d-i} f_{0*} \iota_! \mathbb{Q}_{X_0 \setminus D_0}$ around Σ are unipotent. Then we obtain the following properties.*

- (a) $R^{d-i} f_* \mathcal{O}_X(-D)$ is isomorphic to the lower canonical extension of

$$\mathrm{Gr}_F^0(\mathcal{H}_{Y_0}^{d-i}) = \mathcal{F}^0(\mathcal{H}_{Y_0}^{d-i}) / \mathcal{F}^1(\mathcal{H}_{Y_0}^{d-i}).$$

In particular, $R^{d-i} f_* \mathcal{O}_X(-D)$ is locally free.

By the Grothendieck duality, we obtain

- (b) $R^i f_* \omega_{X/Y}(D)$ is isomorphic to the upper canonical extension of

$$\mathrm{Gr}_F^0(\mathcal{H}_{Y_0}^{d-i})^* = \mathcal{H}om_{\mathcal{O}_{Y_0}}(\mathrm{Gr}_F^0(\mathcal{H}_{Y_0}^{d-i}), \mathcal{O}_{Y_0}).$$

In particular, $R^i f_* \omega_{X/Y}(D)$ is locally free.

Let us start the proof of Theorem 5.1 and Theorem 5.3.

Proof of Theorem 5.1 and Theorem 5.3. The statement (1) in Theorem 5.1 follows from Theorem 4.20 and the arguments in Section 4. We note that (4) in Theorem 5.1 follows from Theorem 6.21 and Corollary 6.23 below by (3) in Theorem 5.1. By [BiP, Theorem 1.2] and Lemma 7.2, we may assume that $\mathrm{Supp}(f^* \Sigma \cup D)$ is a simple normal crossing divisor on X .

In Step 1 and Step 2, we prove (2) and (3) in Theorem 5.1 for every i under the assumption that all the local monodromies around Σ are unipotent (cf. Definition 4.15 and (4.19.4) in Lemma 4.19). The arguments in Step 1 and Step 2 are essentially the same as Nakayama's (cf. [N1, Proof of Theorem 1] and [F4, Section 3]). In Step 3 and Step 4, we prove Theorem 5.3, which contains (2) and (3) in Theorem 5.1.

From now on, we assume that all the local monodromies on all the local systems around Σ are unipotent (cf. Definition 4.15 and (4.19.4) in Lemma 4.19).

Step 1 (The case when $\dim Y = 1$). By shrinking Y , we may assume that Y is the unit disc Δ in \mathbb{C} and $\Sigma = \{0\}$ in Δ . In Example 4.12, by replacing $K_{(X^\bullet, D^\bullet)/Y}^c$ with $K_{(X^\bullet, D^\bullet)/Y}^c(\log 0)$ (cf. Lemma 4.19),

we obtain that $R^{d-i}f_*\mathcal{O}_X(-D)$ is isomorphic to the canonical extension of $\mathrm{Gr}_F^0(\mathcal{H}_{Y_0}^{d-i})$ for every i . Therefore, we obtain $R^if_*\omega_{X/Y}(D) \simeq \mathrm{Gr}_F^0(\mathcal{H}_Y^{d-i})^*$ for every i by the Grothendieck duality.

Step 2 (The case when $l := \dim Y \geq 2$). We shall prove the statement (3) by induction on l for every i .

By Step 1, there is an open subset Y_1 of Y such that $\mathrm{codim}(Y \setminus Y_1) \geq 2$ and that

$$R^if_*\omega_{X/Y}(D)|_{Y_1} \simeq \mathrm{Gr}_F^0(\mathcal{H}_Y^{d-i})^*|_{Y_1}.$$

Since $\mathrm{Gr}_F^0(\mathcal{H}_Y^{d-i})^*$ is locally free, we obtain a homomorphism

$$\varphi_Y^i : R^if_*\omega_{X/Y}(D) \rightarrow \mathrm{Gr}_F^0(\mathcal{H}_Y^{d-i})^*.$$

We will prove that φ_Y^i is an isomorphism. Without loss of generality, we may assume that X and Y are quasi-projective by shrinking Y . By Theorem 7.3 (i), $R^if_*\omega_{X/Y}(D)$ is torsion-free. Therefore, $\mathrm{Ker}\varphi_Y^i = 0$. We put $G_Y^i := \mathrm{Coker}\varphi_Y^i$. Taking a general hyperplane cut, we see that $\mathrm{Supp}G_Y^i$ is a finite set by the induction hypothesis. Assume that $G_Y^i \neq 0$. We may also assume that $\mathrm{Supp}G_Y^i = \{P\}$ by shrinking Y . Let $\mu : W \rightarrow Y$ be the blowing up at P and set $E = \mu^{-1}(P)$. Then $E \simeq \mathbb{P}^{l-1}$. By [BiM, Theorem 1.5] and [BiP, Theorem 1.2], we can take a projective birational morphism $\pi : X' \rightarrow X$ from a simple normal crossing variety X' with the following properties:

- (i) the composition $X' \rightarrow X \rightarrow Y \dashrightarrow W$ is a morphism.
- (ii) π is an isomorphism over X_0 .
- (iii) $\mathrm{Exc}(\pi)$ and $\mathrm{Exc}(\pi) \cup D'$ are simple normal crossing divisors on X' , where D' is the strict transform of D .

We obtain that $R^qf_*\omega_{X/Y}(D) \simeq R^q(f \circ \pi)_*\omega_{X'/Y}(D')$ for every q because $R\pi_*\omega_{X'}(D') \simeq \omega_X(D)$ in the derived category of coherent sheaves on X by Lemma 7.2. We note that every stratum of (X', D') is dominant onto Y . We also note the following commutative diagram.

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ g \downarrow & & \downarrow f \\ W & \xrightarrow[\mu]{} & Y \end{array}$$

By replacing (X, D) with (X', D') , we may assume that there is a morphism $g : X \rightarrow W$ such that $f = \mu \circ g$. Since $g : X \rightarrow W$ is in the same situation as f , we obtain the exact sequence:

$$0 \rightarrow R^ig_*\omega_{X/W}(D) \rightarrow \mathrm{Gr}_F^0(\mathcal{H}_W^{d-i})^* \rightarrow G_W^i \rightarrow 0.$$

Tensoring $\mathcal{O}_W(\nu E)$ for $0 \leq \nu \leq l-1$ and applying $R^j \mu_*$ for $j \geq 0$ to each ν , we have a exact sequence

$$\begin{aligned} 0 &\rightarrow \mu_*(R^i g_* \omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) \rightarrow \mu_*(\mathrm{Gr}_F^0(\mathcal{H}_W^{d-i})^* \otimes \mathcal{O}_W(\nu E)) \\ &\rightarrow \mu_*(G_W^i \otimes \mathcal{O}_W(\nu E)) \rightarrow R^1 \mu_*(R^i g_* \omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) \\ &\rightarrow R^1 \mu_*(\mathrm{Gr}_F^0(\mathcal{H}_W^{d-i})^* \otimes \mathcal{O}_W(\nu E)) \rightarrow 0 \end{aligned}$$

and $R^q \mu_*(R^i g_* \omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) \simeq R^q \mu_*(\mathrm{Gr}_F^0(\mathcal{H}_W^{d-i})^* \otimes \mathcal{O}_W(\nu E))$ for $q \geq 2$.

By [Kw2, Proposition 1], $\mathrm{Gr}_F^0(\mathcal{H}_W^{d-i})^* \simeq \mu^* \mathrm{Gr}_F^0(\mathcal{H}_Y^{d-i})^*$. We have

$$\mu_*(\mathrm{Gr}_F^0(\mathcal{H}_W^{d-i})^* \otimes \mathcal{O}_W(\nu E)) \simeq \mathrm{Gr}_F^0(\mathcal{H}_Y^{d-i})^*$$

and

$$R^q \mu_*(\mathrm{Gr}_F^0(\mathcal{H}_W^{d-i})^* \otimes \mathcal{O}_W(\nu E)) = 0$$

for $q \geq 1$. Therefore, $R^q \mu_*(R^i g_* \omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) = 0$ for $q \geq 2$ and

$$\begin{aligned} 0 &\rightarrow \mu_*(R^i g_* \omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) \rightarrow \mu_*(\mathrm{Gr}_F^0(\mathcal{H}_W^{d-i})^* \otimes \mathcal{O}_W(\nu E)) \\ &\rightarrow \mu_*(G_W^i \otimes \mathcal{O}_W(\nu E)) \rightarrow R^1 \mu_*(R^i g_* \omega_{X/W}(D) \otimes \mathcal{O}_W(\nu E)) \rightarrow 0 \end{aligned}$$

is exact. Since $\omega_W = \mu^* \omega_Y \otimes \mathcal{O}_W((l-1)E)$, we have a spectral sequence

$$E_2^{p,q} = R^p \mu_*(R^q g_* \omega_{X/W}(D) \otimes \mathcal{O}_W((l-1)E)) \Rightarrow R^{p+q} f_* \omega_{X/Y}(D).$$

However, $E_2^{p,q} = 0$ for $p \geq 2$ by the above argument. Thus

$$\begin{aligned} 0 &\rightarrow R^1 \mu_* R^{i-1} g_* \omega_{X/Y}(D) \rightarrow R^i f_* \omega_{X/Y}(D) \\ &\rightarrow \mu_*(R^i g_* \omega_{X/W}(D) \otimes \mathcal{O}_W((l-1)E)) \rightarrow 0. \end{aligned}$$

By Theorem 7.3 (i), $R^i f_* \omega_{X/Y}(D)$ is torsion-free. So, we obtain

$$R^1 \mu_* R^{i-1} g_* \omega_{X/Y}(D) = 0.$$

Therefore, for $q \geq 1$, we obtain

- (a) $R^i f_* \omega_{X/Y}(D) \simeq \mu_*(R^i g_* \omega_{X/W}(D) \otimes \mathcal{O}_W((l-1)E))$ and
- (b) $R^q \mu_*(R^i g_* \omega_{X/W}(D) \otimes \mathcal{O}_W((l-1)E)) = 0$.

Next, we shall consider the following commutative diagram.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
R^i g_* \omega_{X/W}(D) \otimes \mathcal{O}_W((l-2)E) & \rightarrow & R^i g_* \omega_{X/W}(D) \otimes \mathcal{O}_W((l-1)E) \\
\downarrow & & \downarrow \\
\mathrm{Gr}_F^0(\mathcal{H}_W^{d-i})^* \otimes \mathcal{O}_W((l-2)E) & \rightarrow & \mathrm{Gr}_F^0(\mathcal{H}_W^{d-i})^* \otimes \mathcal{O}_W((l-1)E) \\
\downarrow & & \downarrow \\
G_W^i \otimes \mathcal{O}_W((l-2)E) & \rightarrow & G_W^i \otimes \mathcal{O}_W((l-1)E) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

By applying μ_* , we have the next commutative diagram.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mu_*(R^i g_* \omega_{X/W}(D) \otimes \mathcal{O}_W((l-2)E)) & \rightarrow & \mu_*(R^i g_* \omega_{X/W}(D) \otimes \mathcal{O}_W((l-1)E)) \\
\downarrow & & \downarrow \\
\mathrm{Gr}_F^0(\mathcal{H}_Y^{d-i})^* & \simeq & \mathrm{Gr}_F^0(\mathcal{H}_Y^{d-i})^* \\
\downarrow & & \downarrow \\
\mu_*(G_W^i \otimes \mathcal{O}_W((l-2)E)) & \rightarrow & \mu_*(G_W^i \otimes \mathcal{O}_W((l-1)E)) \\
& & \downarrow \\
& & 0
\end{array}$$

By (a) and (b), $G_Y^i \simeq \mu_*(G_W^i \otimes \mathcal{O}_W((l-1)E))$ and

$$\mu_*(G_W^i \otimes \mathcal{O}_W((l-2)E)) \rightarrow \mu_*(G_W^i \otimes \mathcal{O}_W((l-1)E))$$

is surjective. Since $\dim \mathrm{Supp} G_W^i = 0$ and $E \cap \mathrm{Supp} G_W^i \neq \emptyset$, it follows that $G_W^i = 0$ by Nakayama's lemma. Therefore, $G_Y^i = 0$. This implies $R^i f_* \omega_{X/Y}(D) \simeq \mathrm{Gr}_F^0(\mathcal{H}_Y^{d-i})^*$. By the Grothendieck duality, $R^{d-i} f_* \mathcal{O}_X(-D) \simeq \mathrm{Gr}_F^0(\mathcal{H}_Y^{d-i})$.

From now on, we treat the general case, that is, we do not assume that local monodromies are unipotent.

Step 3. In this step, we prove the local freeness of $R^i f_* \omega_{X/Y}(D)$ for every i . We use the unipotent reduction with respect to all the local systems after shrinking Y suitably. Then we obtain the following commutative diagram:

$$\begin{array}{ccccc}
X & \xleftarrow{\alpha} & X' & \xleftarrow{\beta} & \tilde{X} \\
f \downarrow & & f' \downarrow & & \downarrow \tilde{f} \\
Y & \xleftarrow{\tau} & Y' & \xlongequal{\quad} & Y',
\end{array}$$

which satisfies the following properties.

- (i) $\tau : Y' \rightarrow Y$ is a finite Kummer covering from a nonsingular variety Y' and τ ramifies only along Σ .
- (ii) $f' : X' \rightarrow Y'$ is the base change of $f : X \rightarrow Y$ by τ over $Y \setminus \Sigma$.
- (iii) (X', α^*D) is a semi divisorial log terminal pair in the sense of Kollár (cf. [F14, Definition 4.1] and [Ko5, Chapter 4]). Let X_j be any irreducible component of X . Then $X'_j = \alpha^{-1}(X_j)$ is the normalization of the base change of $X_j \rightarrow Y$ by $\tau : Y' \rightarrow Y$ and $X' = \bigcup_j X'_j$. We note that X'_j is a V -manifold for every j . More precisely, X'_j is toroidal for every j .
- (iv) β is a projective birational morphism from a simple normal crossing variety \tilde{X} and \tilde{D} and $\tilde{D} \cup \text{Exc}(\beta)$ are simple normal crossing divisors on \tilde{X} , where \tilde{D} is the strict transform of α^*D (cf. [BiM, Theorem 1.5] and [BiP, Theorem 1.2]). We may further assume that β is an isomorphism over the Zariski open set U of X' , where U is the largest Zariski open set such that (X', α^*D) is a simple normal crossing pair.
- (v) $\tilde{f} : \tilde{X} \rightarrow Y'$, \tilde{D} , and $\tau^{-1}\Sigma$ satisfy the conditions and assumptions in Theorem 5.1 and all the local monodromies on all the local systems around $\tau^{-1}\Sigma$ are unipotent (cf. Definition 4.15 and (4.19.4) in Lemma 4.19).

Therefore, $R^i \tilde{f}_* \omega_{\tilde{X}}(\tilde{D})$ is locally free by Step 1 and Step 2. On the other hand, we can prove

$$R^p \tilde{f}_* \omega_{\tilde{X}}(\tilde{D}) \simeq R^p f'_* \omega_{X'}(\alpha^*D)$$

for every $p \geq 0$. We note that

$$K_{\tilde{X}} + \tilde{D} = \beta^*(K_{X'} + \alpha^*D) + F$$

where F is β -exceptional, F is permissible on \tilde{X} , $\text{Supp} F$ is a simple normal crossing divisor on \tilde{X} , and $\lceil F \rceil$ is effective. Thus we obtain that $\beta_* \omega_{\tilde{X}}(\tilde{D}) \simeq \omega_{X'}(\alpha^*D)$ and that $R^q \beta_* \omega_{\tilde{X}}(\tilde{D}) = 0$ for every $q > 0$ by Lemma 7.1. Thus, $R^i f'_* \omega_{X'}(\alpha^*D)$ is locally free for every i . Since $R^i f_* \omega_X(D)$ is a direct summand of

$$\tau_* R^i f'_* \omega_{X'}(\alpha^*D) \simeq R^i f_*(\alpha_* \omega_{X'}(\alpha^*D)),$$

we obtain that $R^i f_* \omega_X(D)$ is locally free, equivalently, $R^i f_* \omega_{X/Y}(D)$ is locally free for every i . We note that, by the Grothendieck duality, $R^{d-i} f_* \mathcal{O}_X(-D)$ is also locally free for every i .

Step 4. In this last step, we prove that $R^{d-i} f_* \mathcal{O}_X(-D)$ is the lower canonical extension for every i . By the Grothendieck duality and Step 3, $R^{d-i} \tilde{f}_* \mathcal{O}_{\tilde{X}}(-\tilde{D})$ is locally free. By Step 3, we obtain $R \beta_* \omega_{\tilde{X}}(\tilde{D}) \simeq$

$\omega_{X'}(\alpha^*D)$ in the derived category of coherent sheaves on X' . Therefore, we obtain

$$\begin{aligned} R\beta_*\mathcal{O}_{\tilde{X}}(-\tilde{D}) &\simeq R\mathcal{H}om(R\beta_*\omega_{\tilde{X}}^\bullet(\tilde{D}), \omega_{X'}^\bullet) \\ &\simeq R\mathcal{H}om(\omega_{X'}^\bullet(\alpha^*D), \omega_{X'}^\bullet) \simeq \mathcal{O}_{X'}(-\alpha^*D) \end{aligned}$$

in the derived category of coherent sheaves on X' . Note that X' is Cohen–Macaulay (cf. [F14, Theorem 4.2]) and that $\omega_{X'}^\bullet \simeq \omega_{X'}[\dim X']$. Thus, we have

$$R^p\tilde{f}_*\mathcal{O}_{\tilde{X}}(-\tilde{D}) \simeq R^pf'_*\mathcal{O}_{X'}(-\alpha^*D)$$

for every p . Let G be the Galois group of $\tau : Y' \rightarrow Y$. Then we have

$$(\tau_*R^pf'_*\mathcal{O}_{X'}(-\alpha^*D))^G \simeq R^pf_*(\alpha_*\mathcal{O}_{X'}(-\alpha^*D))^G \simeq R^pf_*\mathcal{O}_X(-D).$$

Thus, we obtain that $R^{d-i}f_*\mathcal{O}_X(-D)$ is the lower canonical extension for every i . By the Grothendieck duality, $R^if_*\omega_{X/Y}(D)$ is the upper canonical extension for every i .

We finish the proof of Theorem 5.1 and Theorem 5.3. \square

Remark 5.4. Note that the definition of semi divisorial log terminal pairs in [F1, Definition 1.1] is different from the one in [Ko5, Chapter 4] (cf. [F14, Definition 4.1]).

The following theorem is a generalization of [Ko1, Proposition 7.6].

Theorem 5.5. *Let $f : X \rightarrow Y$ be a projective surjective morphism from a simple normal crossing variety to a smooth algebraic variety Y with connected fibers. Assume that every stratum of X is dominant onto Y . Then $R^df_*\omega_X \simeq \omega_Y$ where $d = \dim X - \dim Y$.*

Proof. By [BiM, Theorem 1.5] and [BiP, Theorem 1.2], we can construct a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\pi} & X \\ g \downarrow & & \downarrow f \\ W & \xrightarrow[p]{} & Y \end{array}$$

with the following properties.

- (i) $p : W \rightarrow Y$ is a projective birational morphism from a smooth quasi-projective variety W .
- (ii) V is a simple normal crossing variety.
- (iii) π is projective birational and π induces an isomorphism $\pi^0 = \pi|_{V^0} : V^0 \rightarrow X^0$ where X^0 (resp. V^0) is a Zariski open set of X (resp. V) which contains the generic point of any stratum of X (resp. V).

- (iv) g is projective.
- (v) there is a simple normal crossing divisor Σ on W such that every stratum of V is smooth over $W \setminus \Sigma$.

We note that $R^j g_* \omega_V$ is locally free for every j by Theorem 5.3. By the Grothendieck duality, we have

$$Rg_* \mathcal{O}_V \simeq R\mathcal{H}om_{\mathcal{O}_W}(Rg_* \omega_V^\bullet, \omega_W^\bullet).$$

Therefore, we have

$$\mathcal{O}_W \simeq \mathcal{H}om_{\mathcal{O}_W}(R^d g_* \omega_V, \omega_W).$$

Note that, by Zariski's main theorem, $g_* \mathcal{O}_V \simeq \mathcal{O}_W$ since every stratum of V is dominant onto W . Thus, we obtain $R^d g_* \omega_V \simeq \omega_W$. By applying p_* , we have $p_* R^d g_* \omega_V \simeq p_* \omega_W \simeq \omega_Y$. We note that $p_* R^d g_* \omega_V \simeq R^d (p \circ g)_* \omega_V$ since $R^i p_* R^d g_* \omega_V = 0$ for every $i > 0$ (cf. Theorem 7.3 (ii)). On the other hand,

$$R^d (p \circ g)_* \omega_V \simeq R^d (f \circ \pi)_* \omega_V \simeq R^d f_* \omega_X$$

since $R^i \pi_* \omega_V = 0$ for every $i > 0$ by Lemma 7.1 and $\pi_* \omega_V \simeq \omega_X$ (cf. Lemma 7.2). Therefore, we obtain $R^d f_* \omega_X \simeq \omega_Y$. \square

We explain a difference between our formulation of Theorems 5.1 and 5.3 and Kawamata's (cf. [Kw3, Theorem 1.1]).

5.6 (Connectedness of fibers). In geometric applications, we sometimes have a projective surjective morphism $f : X \rightarrow Y$ from a simple normal crossing variety to a smooth variety Y with *connected fibers* such that every stratum of X is mapped onto Y . The example below shows that in general there is no stratum S of X such that general fibers of $S \rightarrow Y$ are connected. Therefore, Kawamata's result ([Kw3, Theorem 1.1]) is very restrictive. He assumes that $S \rightarrow Y$ has connected fibers for every stratum S of X .

Example 5.7. We consider $W = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Let $p_i : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the i -th projection for $i = 1, 2, 3$. We take general members $X_1 \in |p_1^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(2)|$ and $X_2 \in |p_1^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_3^* \mathcal{O}_{\mathbb{P}^1}(2)|$. We define $X = X_1 \cup X_2$, $Y = \mathbb{P}^1$, and $f = p_1|_X : X \rightarrow Y$. Then f is a projective morphism from a simple normal crossing variety X to a smooth projective curve Y . We can directly check that

$$H^1(W, \mathcal{O}_W(-X_1)) = H^1(W, \mathcal{O}_W(-X_2)) = 0$$

and

$$H^1(W, \mathcal{O}_W(-X_1 - X_2)) = H^2(W, \mathcal{O}_W(-X_1 - X_2)) = 0.$$

Therefore, by using

$$0 \rightarrow \mathcal{O}_W(-X_1 - X_2) \rightarrow \mathcal{O}_W(-X_2) \rightarrow \mathcal{O}_{X_1}(-X_2) \rightarrow 0,$$

we obtain $H^1(X_1, \mathcal{O}_{X_1}(-X_2)) = 0$. By using

$$0 \rightarrow \mathcal{O}_{X_1}(-X_2) \rightarrow \mathcal{O}_{X_1} \rightarrow \mathcal{O}_{X_1 \cap X_2} \rightarrow 0,$$

we obtain $H^0(X_1 \cap X_2, \mathcal{O}_{X_1 \cap X_2}) = \mathbb{C}$ since $H^0(X_1, \mathcal{O}_{X_1}) = \mathbb{C}$. This means that $C = X_1 \cap X_2$ is a smooth connected curve. Therefore, every stratum of X is mapped onto Y by f . We note that general fibers of $f : X_1 \rightarrow Y$, $f : X_2 \rightarrow Y$, and $f : C \rightarrow Y$ are disconnected.

As a special case of Theorem 5.1, we obtain the following theorem (cf. [Kw1, Theorem 5], [Kw2, Theorem 2], [Ko2, Theorem 2.6], [N1, Theorem 1]).

Theorem 5.8. *Let $f : X \rightarrow Y$ be a projective morphism between smooth complete algebraic varieties which satisfies the following conditions:*

- (i) *There is a Zariski open subset Y_0 of Y such that $\Sigma = Y \setminus Y_0$ is a simple normal crossing divisor on Y .*
- (ii) *We put $X_0 = f^{-1}(Y_0)$ and $f_0 = f|_{X_0}$. Then f_0 is smooth.*
- (iii) *The local monodromies of $R^{d+i}f_{0*}\mathbb{C}_{X_0}$ around Σ are unipotent, where $d = \dim X - \dim Y$.*

Then $R^i f_ \omega_{X/Y}$ is a semi-positive locally free sheaf on Y .*

Proof. By the Poincaré–Verdier duality (see, for example, [PS, Theorem 13.9]), $R^{d-i}f_{0*}\mathbb{C}_{X_0}$ is the dual local system of $R^{d+i}f_{0*}\mathbb{C}_{X_0}$. Therefore, the local monodromies of $R^{d-i}f_{0*}\mathbb{C}_{X_0}$ around Σ are unipotent. Thus, by Theorem 5.1, we obtain that $R^i f_* \omega_{X/Y} \simeq (R^{d-i}f_* \mathcal{O}_X)^*$ is a semi-positive locally free sheaf on Y . \square

Similarly, the semi-positivity theorem in [F4] (cf. [F4, Theorem 3.9]) can be recovered by Theorem 5.1. We note that [Kw3, Theorem 1.1] does not cover [F4, Theorem 3.9]. It is because Kawamata’s theorem needs that $S \rightarrow Y$ has connected fibers for every stratum S of (X, D) (cf. 5.6).

Remark 5.9. Theorem 5.8 was first proved by Kawamata (cf. [Kw1, Theorem 5]) with the extra assumptions that $i = 0$ and that f has connected fibers. The above statement is a combination of [Kw2, Theorem 2] with [Ko2, Theorem 2.6] or [N1, Theorem 1].

Remark 5.10. Let $f : X \rightarrow Y$ be a projective morphism between smooth projective varieties. Assume that there exists a simple normal crossing divisor Σ on Y such that f is smooth over $Y \setminus \Sigma$. Then

$R^i f_* \omega_{X/Y}$ is locally free for every i (cf. Theorem 5.3 and [Ko2, Theorem 2.6]). We note that $R^i f_* \omega_{X/Y}$ is not always semi-positive if we assume nothing on monodromies around Σ .

We close this section with an easy example.

Example 5.11 (Double cover). We consider $\pi : Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \rightarrow \mathbb{P}^1$. Let E and G be the sections of π such that $E^2 = -2$ and $G^2 = 2$. We note that $E + 2F \sim G$ where F is a fiber of π . We put $\mathcal{L} = \mathcal{O}_Y(E + F)$. Then $E + G \in |\mathcal{L}^{\otimes 2}|$. Let $f : X \rightarrow Y$ be the double cover constructed by $E + G \in |\mathcal{L}^{\otimes 2}|$. Then $f : X \rightarrow Y$ is étale outside $\Sigma = E + G$ and

$$f_* \omega_{X/Y} \simeq \mathcal{O}_Y \oplus \mathcal{L}.$$

In this case, $f_* \omega_{X/Y}$ is not semi-positive since $\mathcal{L} \cdot E = -1$. We note that the local monodromies on $f_{0*} \mathbb{C}_{X_0}$ around Σ are not unipotent, where $Y_0 = Y \setminus \Sigma$, $X_0 = f^{-1}(Y_0)$, and $f_0 = f|_{X_0}$.

In Example 5.11, $f : X \rightarrow Y$ is finite and the general fibers of f are disconnected. In Section 8, we discuss an example $f : X \rightarrow Y$ whose general fibers are elliptic curves such that $f_* \omega_{X/Y}$ is not semi-positive (cf. Corollary 8.10 and Example 8.16).

6. SEMI-POSITIVITY THEOREM

In this section, we discuss a purely Hodge theoretic aspect of the Fujita–Kawamata semi-positivity theorem (cf. [Kw1, §4 Semi-positivity]). Our formulation is different from Kawamata’s original one but is indispensable for our main theorem: Theorem 5.1 (4). For related topics, see [Mo, Section 5], [F5, Section 5], [F4, 3.2. Semi-positivity theorem], and [Ko4, 8.10]. We use the theory of logarithmic integrable connections. For the basic properties and results on logarithmic integrable connections, see [D1], [Kt], and [Bo, IV. Regular connections, after Deligne] by Bernard Malgrange.

We start with easy observations.

Lemma 6.1. *Let X be a complex manifold, U a dense open subset of X and \mathcal{V} a locally free \mathcal{O}_X -module of finite rank. Assume that two \mathcal{O}_X -submodules \mathcal{F} and \mathcal{G} satisfy the following conditions:*

(6.1.1) \mathcal{G} and $\mathcal{V} / \mathcal{G}$ are locally free \mathcal{O}_X -modules of finite rank.

(6.1.2) $\mathcal{F}|_U = \mathcal{G}|_U$.

Then we have the inclusion $\mathcal{F} \subset \mathcal{G}$ on X .

Corollary 6.2. *Let X , U and \mathcal{V} be as above. Two finite decreasing filtrations F and G on \mathcal{V} satisfy the following conditions:*

- $\mathrm{Gr}_G^p \mathcal{V}$ is a locally free \mathcal{O}_X -module of finite rank for every p .
- $F^p \mathcal{V}|_U = G^p \mathcal{V}|_U$ for every p .

Then we have $F^p \mathcal{V} \subset G^p \mathcal{V}$ on X for every p . In particular, $F^p \mathcal{V} = G^p \mathcal{V}$ for every p , if, in addition, $\mathrm{Gr}_F^p \mathcal{V}$ is locally free of finite rank for every p .

6.3. Let X be a complex manifold and $D = \sum_{i \in I} D_i$ a simple normal crossing divisor on X , where D_i is a smooth irreducible divisor on X for every $i \in I$. We set

$$D(J) = \bigcap_{i \in J} D_i, \quad D_J = \sum_{i \in J} D_i$$

for any subset J . Note that $D(\emptyset) = X$ and $D_\emptyset = 0$ by definition. Moreover we set $D(J)^* = D(J) \setminus D(J) \cap D_{I \setminus J}$ for $J \subset I$. For the case of $J = \emptyset$, we set $X^* = D(\emptyset)^* = X \setminus D$.

Let \mathcal{V} be a locally free \mathcal{O}_X -module of finite rank and

$$\nabla : \mathcal{V} \longrightarrow \Omega_X^1(\log D) \otimes \mathcal{V}$$

a logarithmic integrable connection on \mathcal{V} . We denote by

$$\mathrm{Res}_{D_i}(\nabla) : \mathcal{O}_{D_i} \otimes \mathcal{V} \longrightarrow \mathcal{O}_{D_i} \otimes \mathcal{V}$$

the residue of ∇ along D_i . We assume the following condition throughout this section:

$$(6.3.1) \quad \mathrm{Res}_{D_i}(\nabla) : \mathcal{O}_{D_i} \otimes \mathcal{V} \longrightarrow \mathcal{O}_{D_i} \otimes \mathcal{V} \text{ is nilpotent for every } i \in I.$$

This is equivalent to the condition that the local system $\mathrm{Ker}(\nabla)|_{X^*}$ is of unipotent monodromy.

6.4. In the situation above, we denote the morphism

$$\mathrm{id} \otimes \mathrm{Res}_{D(J)}(\nabla) : \mathcal{O}_{D(J)} \otimes \mathcal{V} \longrightarrow \mathcal{O}_{D(J)} \otimes \mathcal{V}$$

by $N_{i,D(J)}$ for a subset J of I and for $i \in J$. We simply denote by N_i if there is no danger of confusion. We have

$$N_{i,D(J)} N_{j,D(J)} = N_{j,D(J)} N_{i,D(J)}$$

for every $i, j \in J$. For two subsets J, K of I with $K \subset J$, we set $N_{K,D(J)} = \sum_{i \in K} N_{i,D(J)}$, which is nilpotent by the assumption above. Once a subset J is fixed, we use the symbols N_K for short. We have the monodromy weight filtration $W(K)$ on $\mathcal{O}_{D(J)} \otimes \mathcal{V}$ which is characterized by the condition that N_K^q induces an isomorphism

$$\mathrm{Gr}_q^{W(K)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \xrightarrow{\simeq} \mathrm{Gr}_{-q}^{W(K)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$$

for all $q \geq 0$. For $K = \emptyset$, $W(\emptyset)$ is trivial, that is, $W(\emptyset)_{-1} \mathcal{O}_{D(J)} \otimes \mathcal{V} = 0$ and $W(\emptyset)_0 \mathcal{O}_{D(J)} \otimes \mathcal{V} = \mathcal{O}_{D(J)} \otimes \mathcal{V}$.

For the case of $J = K$, we set

$$\mathcal{P}_k(J) = \text{Ker}(N_J^{k+1} : \text{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \longrightarrow \text{Gr}_{-k-2}^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}))$$

for every non-negative integer k , which is called the primitive part of $\text{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$ with respect to N_J . Then we have the primitive decomposition

$$\text{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) = \bigoplus_{l \geq \max(0, -k)} N_J^l(\mathcal{P}_{k+2l}(J))$$

for every k , and N_J^l induces an isomorphism

$$\mathcal{P}_{k+2l}(J) \longrightarrow N_J^l(\mathcal{P}_{k+2l}(J))$$

for every k, l with $l \geq \max(0, -k)$.

Lemma 6.5. *In the situation above, $\text{Gr}_k^{W(K)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$ is a locally free $\mathcal{O}_{D(J)}$ -module of finite rank for every k and for every subsets J, K of I with $K \subset J$.*

Proof. Easy by the local description of logarithmic integrable connection (see e.g. Deligne [D1], Katz [Kt]). \square

Corollary 6.6. *In the situation above, we fix a subset J of I . For any subset K of J we have the equality*

$$W(K) = W(N_K(x))$$

on $\mathcal{V}(x) = \mathcal{V} \otimes \mathbb{C}(x)$ for every point $x \in D(J)$, where the left hand side denotes the filtration on $\mathcal{V}(x)$ induced by $W(K)$.

Remark 6.7. Let $(\mathcal{V}_1, \nabla_1)$ and $(\mathcal{V}_2, \nabla_2)$ be pairs of locally free sheaves of \mathcal{O}_X -modules of finite rank and integrable logarithmic connections on them. We assume that they satisfy the condition in 6.3. If the morphism $\varphi : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ of \mathcal{O}_X -modules is compatible with the connections ∇_1 and ∇_2 , then the diagram

$$\begin{array}{ccc} \mathcal{O}_{D(J)} \otimes \mathcal{V}_1 & \xrightarrow{\text{id} \otimes \varphi} & \mathcal{O}_{D(J)} \otimes \mathcal{V}_2 \\ N_{i,D(J)} \downarrow & & \downarrow N_{i,D(J)} \\ \mathcal{O}_{D(J)} \otimes \mathcal{V}_1 & \xrightarrow{\text{id} \otimes \varphi} & \mathcal{O}_{D(J)} \otimes \mathcal{V}_2 \end{array}$$

is commutative for every subset J of I and for every $i \in J$. Therefore $\text{id} \otimes \varphi$ preserves the filtration $W(K)$ for every $K \subset J$.

6.8. Let m be an integer. For a finite decreasing filtration F on \mathcal{V} , we consider the following condition:

(mMH) The triple

$$(\mathcal{V}(x), W(J)[m], F)$$

underlies an \mathbb{R} -mixed Hodge structure for any subset J of I and for any point $x \in D(J)^*$.

Here we remark that we do not assume the local freeness of $\mathrm{Gr}_F^p \mathcal{V}$ at the beginning.

The following lemma is the counterpart of Schmid's results in [Sc].

Lemma 6.9. *Let U be an open subset of $X \setminus D$, such that $X \setminus U$ is nowhere dense analytic subspace of X . Moreover, we are given a finite decreasing filtration F on $\mathcal{V}|_U$. If $(\mathcal{V}|_U, F, \nabla)$ underlies a polarizable variation of \mathbb{R} -Hodge structure of weight m on U , then there exists a finite decreasing filtration \tilde{F} on \mathcal{V} satisfying the following three conditions:*

- (1) $\tilde{F}^p \mathcal{V}|_U = F^p \mathcal{V}|_U$ for every p .
- (2) $\mathrm{Gr}_{\tilde{F}}^p \mathcal{V}$ is a locally free \mathcal{O}_X -module of finite rank for every p .
- (3) \tilde{F} satisfies the condition (mMH) in 6.8.

Proof. See [Sc]. □

Lemma 6.10. *Let U be as above, and F a finite decreasing filtration on \mathcal{V} in the situation 6.3. We assume that $(\mathcal{V}, F, \nabla)|_U$ underlies a polarizable variation of \mathbb{R} -Hodge structure of weight m on U . Then $\mathrm{Gr}_F^p \mathcal{V}$ is locally free of finite rank for every p if and only if F satisfies the condition (mMH) in 6.8.*

Proof. By the lemma above, there exists a finite decreasing filtration \tilde{F} on \mathcal{V} satisfying the three conditions above. By Corollary 6.2, the local freeness of $\mathrm{Gr}_F^p \mathcal{V}$ for every p is equivalent to the equality $F^p \mathcal{V} = \tilde{F}^p \mathcal{V}$ for every p . If $F = \tilde{F}$ on \mathcal{V} , F satisfies the condition (mMH) by the lemma above. Thus it suffices to prove the equality $F = \tilde{F}$ on \mathcal{V} under the assumption that F satisfies the condition (mMH). By Corollary 6.2 again, we have $F^p \mathcal{V} \subset \tilde{F}^p \mathcal{V}$ for every p . On the other hand, $(\mathcal{V}(x), W(J)[m], F)$ and $(\mathcal{V}(x), W(J)[m], \tilde{F})$ are \mathbb{R} -mixed Hodge structures for every $x \in D(J)^*$, if F satisfies the condition (mMH). Therefore we obtain $F(\mathcal{V}(x)) = \tilde{F}(\mathcal{V}(x))$ for every $x \in X$, which implies the equality $F = \tilde{F}$ on \mathcal{V} . □

6.11. In addition to the situation 6.3, we assume that we are given a finite decreasing filtration F on \mathcal{V} satisfying the following three conditions:

- The Griffiths transversality holds, that is, we have $\nabla(F^p) \subset \Omega_X^1(\log D) \otimes F^{p-1}$ for every p .
- $(\mathcal{V}, F, \nabla)|_{X^*}$ underlies a polarizable variation of \mathbb{R} -Hodge structure of weight m .
- $\mathrm{Gr}_F^p \mathcal{V}$ is locally free of finite rank for every p , or equivalently, F satisfies the condition $(m\mathrm{MH})$.

For a subset J of I , the Griffiths transversality implies the condition

$$N_i(F^p(\mathcal{O}_{D(J)} \otimes \mathcal{V})) \subset F^{p-1}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$$

for every p and for every $i \in J$.

Lemma 6.12. *In the situation above, we have*

- (1) $N_i(W(K)_k) \subset W(K)_{k-1}$ for every $i \in K$ and for every k ,
- (2) $W(J)$ is the monodromy weight filtration of N_K relative to the filtration $W(J \setminus K)$

on $\mathcal{O}_{D(J)} \otimes \mathcal{V}$ for every two subsets J, K of I with $K \subset J$.

Proof. See Cattani–Kaplan [CK, (3.3) Theorem, (3,4)] and Steenbrink–Zucker [SZ, (3.12) Theorem]. \square

Corollary 6.13. *In the situation 6.3 and 6.11, the induced filtration F on $\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$ satisfies the property $((m+k)\mathrm{MH})$ for any subset J of I .*

Proof. Take a subset K of $I \setminus J$. For any point $x \in D(J \cup K)^*$, the triple

$$(\mathcal{V}(x), W(J \cup K)[m], F)$$

underlies an \mathbb{R} -mixed Hodge structure because F satisfies the condition $(m\mathrm{MH})$ by the assumption. Moreover, the morphism $(2\pi\sqrt{-1})^{-1}N_J(x)$ is a morphism of \mathbb{R} -mixed Hodge structures of type $(-1, -1)$ by the condition (2) in the lemma above and by the Griffiths transversality. Therefore

$$(\mathrm{Gr}_k^{W(J)} \mathcal{V}(x), W(J \cup K)[m], F)$$

is an \mathbb{R} -mixed Hodge structure. On the other hand, we have

$$W(J \cup K)(\mathrm{Gr}_k^{W(J)} \mathcal{V}(x)) = W(K)(\mathrm{Gr}_k^{W(J)} \mathcal{V}(x))[k],$$

by (2) in the lemma above. Thus

$$(\mathrm{Gr}_k^{W(J)} \mathcal{V}(x), W(K)[m+k], F)$$

underlies an \mathbb{R} -mixed Hodge structure. \square

6.14. In the situation 6.3 and 6.11 we fix a subset J of I . We have an exact sequence

$$\begin{aligned} 0 \longrightarrow \Omega_{D(J)}^1(\log D(J) \cap D_{I \setminus J}) &\longrightarrow \Omega_X^1(\log D) \otimes \mathcal{O}_{D(J)} \\ &\longrightarrow \mathcal{O}_{D(J)}^{\oplus |J|} \longrightarrow 0, \end{aligned}$$

where $|J|$ denotes the cardinality of J . On the other hand, the integrable log connection ∇ induces a commutative diagram

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\nabla} & \Omega_X^1(\log D) \otimes \mathcal{V} \\ \downarrow & & \downarrow \\ \mathcal{O}_{D(J)} \otimes \mathcal{V} & \longrightarrow & \mathcal{O}_{D(J)}^{\oplus |J|} \otimes \mathcal{V}, \end{array}$$

where the bottom horizontal arrow coincides with $\bigoplus_{i \in J} N_{i, D(J)}$ under the identification $\mathcal{O}_{D(J)}^{\oplus |J|} \otimes \mathcal{V} \simeq (\mathcal{O}_{D(J)} \otimes \mathcal{V})^{|J|}$. Because ∇ preserves the filtration $W(J)$ on $\mathcal{O}_{D(J)} \otimes \mathcal{V}$ by the local description in [D1], [Kt] and because $N_{i, D(J)}(W(J)_k) \subset W(J)_{k-1}$ for every k by Lemma 6.12 (1), we obtain a morphism

$$\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \longrightarrow \Omega_{D(J)}^1(\log D(J) \cap D_{I \setminus J}) \otimes \mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$$

for every k . We denote it by $\nabla_k(J)$, or simply $\nabla(J)$. It is easy to see that $\nabla(J)$ is an integrable log connection on $\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$ satisfying $\nabla(J)(F^p) \subset F^{p-1}$ for every p for the induced filtration F on $\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$. We can easily see that the residue of $\nabla(J)$ along $D(J) \cap D_i$ coincide with $N_{i, D(J \cup \{i\})}$ for $i \in I \setminus J$. Thus $\nabla(J)$ satisfies the condition in 6.3.

6.15. Let (\mathcal{V}, F, ∇) be as in 6.3 and 6.11. Then $(\mathcal{V}, F, \nabla)|_{X^*}$ is a polarizable variation of \mathbb{R} -Hodge structure of weight m . An integrable logarithmic connection on $\mathcal{V} \otimes \mathcal{V}$ is defined by $\nabla \otimes \mathrm{id} + \mathrm{id} \otimes \nabla$ as usual. Assume that we are given a morphism

$$S : \mathcal{V} \otimes \mathcal{V} \longrightarrow \mathcal{O}_X$$

satisfying the following:

- S is $(-1)^m$ -symmetric.
- S is compatible with the connections, where \mathcal{O}_X is equipped with the trivial connection d .
- $S(F^p \mathcal{V} \otimes F^q \mathcal{V}) = 0$ if $p + q > m$.
- $S|_{X^*}$ underlies a polarization of the variation of \mathbb{R} -Hodge structure $(\mathcal{V}, F, \nabla)|_{X^*}$.

Now we fix a subset J of I . Then S induces a morphism

$$\mathcal{O}_{D(J)} \otimes \mathcal{V} \otimes \mathcal{V} \simeq (\mathcal{O}_{D(J)} \otimes \mathcal{V}) \otimes (\mathcal{O}_{D(J)} \otimes \mathcal{V}) \longrightarrow \mathcal{O}_{D(J)},$$

which is denoted by S_J .

Lemma 6.16. *In the situation above, we have*

$$S_J(W(K)_a \otimes W(K)_b) = 0$$

for every $K \subset J$ and for every a, b with $a + b < 0$.

Proof. We fix a subset K of J . It is sufficient to prove that

$$S_J(W(K)_a \otimes W(K)_{-a-1}) = 0$$

for every non-negative integer a .

Since S is compatible with the connections, we have

$$S_J \cdot (N_i \otimes \text{id} + \text{id} \otimes N_i) = 0$$

for every $i \in J$, from which the equality

$$S_J \cdot (N_K \otimes \text{id} + \text{id} \otimes N_K) = 0$$

is obtained. Then we have

$$\begin{aligned} S_J(W(K)_a \otimes W(K)_{-a-1}) &= (S_J \cdot \text{id} \otimes N_K^{a+1})(W(K)_a \otimes W(K)_{a+1}) \\ &= (-1)^{a+1}(S_J \cdot N_K^{a+1} \otimes \text{id})(W(K)_a \otimes W(K)_{a+1}) \\ &= (-1)^{a+1}S_J(W(K)_{-a-2} \otimes W(K)_{a+1}) \\ &= (-1)^{a+1+m}S_J(W(K)_{a+1} \otimes W(K)_{-a-2}) \end{aligned}$$

by using the equality $W(K)_{-k} = N^k(W(K)_k)$ for $k \geq 0$ (see e.g. [SZ, (2.2) Corollary]). Thus we obtain the conclusion by descending induction on a . \square

Corollary 6.17. *In the situation above, S_J induces a morphism*

$$\text{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \otimes \text{Gr}_{-k}^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \longrightarrow \mathcal{O}_{D(J)}$$

for a non-negative integer k .

6.18. In the situation above, we define a morphism

$$\overline{S}_k(J) : \mathcal{P}_k(J) \otimes \mathcal{P}_k(J) \longrightarrow \mathcal{O}_{D(J)}$$

by $\overline{S}_k(J) = S_J \cdot (\text{id} \otimes N_J^k)$ for every subset $J \subset I$ and for every non-negative integer k .

By using the direct sum decomposition

$$\text{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) = \bigoplus_{l \geq 0} N^l(\mathcal{P}_{k+2l}(J))$$

for a non-negative integer k , we obtain a morphism

$$S_k(J) : \mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \otimes \mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \longrightarrow \mathcal{O}_{D(J)}$$

which is characterized by the following properties:

- For non-negative integers a, b we have

$$S_k(J)(N^a(\mathcal{P}_{k+2a}(J)) \otimes N^b(\mathcal{P}_{k+2b}(J))) = 0$$

if $a \neq b$.

- The diagram

$$\begin{array}{ccc} \mathcal{P}_{k+2l}(J) \otimes \mathcal{P}_{k+2l}(J) & \xrightarrow{\overline{S}_{k+2l}(J)} & \mathcal{O}_{D(J)} \\ N^l \otimes N^l \downarrow & & \parallel \\ N^l(\mathcal{P}_{k+2l}(J)) \otimes N^l(\mathcal{P}_{k+2l}(J)) & \xrightarrow{S_k(J)} & \mathcal{O}_{D(J)} \end{array}$$

is commutative for every non-negative integer l .

For a positive integer k , the morphism

$$S_{-k}(J) : \mathrm{Gr}_{-k}^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \otimes \mathrm{Gr}_{-k}^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \longrightarrow \mathcal{O}_{D(J)}$$

is defined by identifying $\mathrm{Gr}_{-k}^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$ with $\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$ via the morphism $N(J)^k$. More precisely, $S_{-k}(J)$ is the unique morphism such that the diagram

$$\begin{array}{ccc} \mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \otimes \mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) & \xrightarrow{S_k(J)} & \mathcal{O}_{D(J)} \\ N(J)^k \otimes N(J)^k \downarrow & & \parallel \\ \mathrm{Gr}_{-k}^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \otimes \mathrm{Gr}_{-k}^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) & \xrightarrow{S_{-k}(J)} & \mathcal{O}_{D(J)} \end{array}$$

is commutative.

The following proposition plays the key role in the proof of semi-positivity theorem.

Proposition 6.19. *In the situation 6.3, 6.11 and 6.15, the data*

$$(\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}), F, \nabla(J), S_k(J))$$

satisfies the conditions in 6.3, 6.11 and 6.15 again.

Proof. By Lemma 6.5 and 6.14, the pair $(\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}), \nabla(J))$ satisfies the condition 6.3. As remarked in 6.14, we have $\nabla(J)(F^p) \subset F^{p-1}$ for every p . Moreover, the filtration F on $\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$ satisfies the condition $((m+k)\mathrm{MH})$ by Corollary 6.13.

By definition, the morphism

$$S_J : \mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \otimes \mathrm{Gr}_{-k}^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$$

is compatible with the connections on both sides. Therefore $\overline{S}_k(J)$ is compatible with the connections because

$$N_J : \mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}) \longrightarrow \mathrm{Gr}_{k-2}^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$$

is compatible with the connection $\nabla(J)$ on the both sides. Thus $S_k(J)$ is compatible with the connection. Moreover we can check the equality

$$S_k(J)(F^p \otimes F^q) = 0$$

for $p + q > m + k$ by using $N_J^k(F^q) \subset F^{q-k}$.

There exists an open subset U of $D(J)^*$ such that $\mathrm{Gr}_F^p \mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$ is locally free $\mathcal{O}_{D(J)}$ -module of finite rank for every p and that $D(J) \setminus U$ is a nowhere dense closed analytic subspace of $D(J)$.

By the local description as in Deligne [D1], Katz [Kt], we can easily check that $\mathrm{Ker}(\nabla_k(J))|_{D(J)^*}$ admits an \mathbb{R} -structure, that is, there exists a local system $\mathbb{V}_k(J)$ of finite dimensional \mathbb{R} -vector spaces with the property $\mathbb{C} \otimes \mathbb{V}_k(J) \simeq \mathrm{Ker}(\nabla_k(J))|_{D(J)^*}$. Then the data

$$(\mathbb{V}_k(J), (\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}), F), \nabla(J), S_k(J))|_U$$

is a polarized variation of \mathbb{R} -Hodge structure of weight $m+k$, by Schmid [Sc]. By Lemma 6.10, $\mathrm{Gr}_F^p \mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V})$ turns out to be locally free for every k, p and then

$$(\mathrm{Gr}_k^{W(J)}(\mathcal{O}_{D(J)} \otimes \mathcal{V}), F, \nabla(J), S_k(J))|_{D(J)^*}$$

underlies a polarized variation of \mathbb{R} -Hodge structure of weight $m+k$ as desired. By the continuity $S_k(J)$ is $(-1)^{m+k}$ -symmetric. \square

Let us recall the definition of semi-positive vector bundles in the sense of Fujita–Kawamata. Example 8.2 below helps us understand the Fujita–Kawamata semi-positivity.

Definition 6.20 (Semi-positivity). A locally free sheaf (or a vector bundle) \mathcal{E} on a complete algebraic variety X is said to be *semi-positive* if for every smooth curve C , for every morphism $\varphi : C \rightarrow X$, and for every quotient invertible sheaf (or line bundle) \mathcal{Q} of $\varphi^*\mathcal{E}$, we have $\deg_C \mathcal{Q} \geq 0$.

It is easy to see that \mathcal{E} is semi-positive if and only if $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ is nef where $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ is the tautological line bundle on $\mathbb{P}_X(\mathcal{E})$.

The following theorem is the main result of this section (cf. [Kw1, Theorem 5]). It is a completely Hodge theoretic result.

Theorem 6.21 (Semi-positivity theorem). *Let X be a smooth complete complex variety, D a simple normal crossing divisor on X , \mathcal{V} a locally free \mathcal{O}_X -module of finite rank equipped with a finite increasing filtration W and a finite decreasing filtration F . We assume the following:*

- (1) $F^a \mathcal{V} = \mathcal{V}$ and $F^{b+1} \mathcal{V} = 0$ for some $a < b$.
- (2) $\mathrm{Gr}_F^p \mathrm{Gr}_m^W \mathcal{V}$ is a locally free \mathcal{O}_X -module of finite rank for all m, p .
- (3) For all m , $\mathrm{Gr}_m^W \mathcal{V}$ admits an integrable logarithmic connection ∇_m with the nilpotent residue morphisms which satisfies the conditions $\nabla_m(F^p \mathrm{Gr}_m^W \mathcal{V}) \subset F^{p-1} \mathrm{Gr}_m^W \mathcal{V}$ for all p .
- (4) The pair $(\mathrm{Gr}_m^W \mathcal{V}, F, \nabla_m)|_{X \setminus D}$ underlies a polarizable variation of \mathbb{R} -Hodge structure of weight m for every integer m .

Then $(\mathrm{Gr}_F^a \mathcal{V})^*$ and $F^b \mathcal{V}$ are semi-positive.

Proof. Since a vector bundle which is an extension of two semi-positive vector bundles is also semi-positive, we may assume that the given \mathcal{V} is pure of weight m , that is, $W_m \mathcal{V} = \mathcal{V}, W_{m-1} \mathcal{V} = 0$, for an integer m without loss of generality. Then \mathcal{V} carries an integrable logarithmic connection ∇ whose residue morphisms are nilpotent. Thus the data (\mathcal{V}, F, ∇) satisfies the conditions in 6.3 and 6.11. Note that \mathcal{V} is the canonical extension of $\mathcal{V}|_{X \setminus D}$ because the residue morphisms of ∇ are nilpotent.

By the assumption (4) above, $\mathcal{V}|_{X \setminus D}$ carries a polarization which extends to a morphism

$$S : \mathcal{V} \otimes \mathcal{V} \longrightarrow \mathcal{O}_X$$

by functoriality of the canonical extensions. We can easily see that the data (\mathcal{V}, F, ∇) and S satisfies the conditions in 6.3, 6.11 and 6.15.

For the case of $\dim X = 1$, we obtain the conclusion by Zucker [Z] (see also Kawamata [Kw1] and the proof of [Ko3, Theorem 5.20]).

Next, we study the case of $\dim X > 1$. Let $\varphi : C \longrightarrow X$ be a morphism from a smooth projective curve. The irreducible decomposition of D is denoted by $D = \sum_{i \in I} D_i$ as in 6.3. We set $J = \{i \in I; \varphi(C) \subset D_i\} \subset I$. Then $\varphi(C) \subset D(J)$, $\varphi(C) \cap D(J)^* \neq \emptyset$ and $\varphi^* D_{I \setminus J}$ is an effective divisor on C . By Proposition 6.19, the locally free sheaf $\mathcal{O}_{D(J)} \otimes \mathcal{V}$ with the finite increasing filtration $W(J)$ and the finite decreasing filtration F satisfies the assumptions (1)-(4) for $D(J)$ with the simple normal crossing divisor $D(J) \cap D_{I \setminus J}$. Therefore $\varphi^* \mathcal{V} = \varphi^*(\mathcal{O}_{D(J)} \otimes \mathcal{V})$ with the induced filtrations W and F satisfies the assumptions (1)-(4) for C with the effective divisor $\varphi^* D_{I \setminus J}$. Then we conclude the desired semi-positivity by the case of $\dim X = 1$. \square

Remark 6.22. In Theorem 6.21, if X is not complete, then we have the following statement. Let V be a complete subvariety of X . Then

$(\mathrm{Gr}_F^a \mathcal{V})^*|_V$ and $(F^b \mathcal{V})|_V$ are semi-positive locally free sheaves on V . It is obvious by the proof of Theorem 6.21.

Corollary 6.23. *Let X and D be as in Theorem 6.21. Assume that we are given an admissible graded polarizable variation of \mathbb{R} -mixed Hodge structure $V = ((\mathbb{V}, W), F)$ on $X \setminus D$ of unipotent monodromy. We assume the conditions $F^a \mathcal{V} = \mathcal{V}$ and $F^{b+1} \mathcal{V} = 0$. The canonical extensions of $\mathcal{V} = \mathcal{O}_{X \setminus D} \otimes \mathbb{V}$ and of $W_k \mathcal{V} = \mathcal{O}_{X \setminus D} \otimes W_k$ are denoted by $\tilde{\mathcal{V}}$ and by $W_k \tilde{\mathcal{V}}$ for all k . As stated in Proposition 3.13, the Hodge filtration F extends to $\tilde{\mathcal{V}}$ such that $\mathrm{Gr}_F^p \mathrm{Gr}_k^W \tilde{\mathcal{V}}$ is locally free of finite rank for all k, p . Then $(\mathrm{Gr}_F^a \tilde{\mathcal{V}})^*$ and $F^b \tilde{\mathcal{V}}$ are semi-positive.*

We learned the following remark from Christopher Hacon.

Remark 6.24. The proof of the semi-positivity theorem in [Ko4, Theorem 8.10.12] contains some ambiguities. In the same notation as in [Ko4, Theorem 8.10.12], if D is a simple normal crossing divisor but is not a *smooth* divisor, then it is not clear how to express $R^m f_* \omega_{X/Y}(D)$ as an extension of $R^m f_* \omega_{D_j/Y}$'s. The case when $D = F$ is a *smooth* divisor is treated in the proof of [Ko4, Theorem 8.10.12]. The same argument does not seem to be sufficient for the general case.

Fortunately, [F4, Theorem 3.9] is sufficient for all applications in [Ko4].

7. VANISHING AND TORSION-FREE THEOREMS

In this section, we discuss some generalizations of torsion-free and vanishing theorems for *quasi-projective* simple normal crossing pairs.

First, let us recall the following very useful lemma. For a proof, see, for example, [F14, Lemma 3.2].

Lemma 7.1 (Relative vanishing lemma). *Let $f : Y \rightarrow X$ be a proper morphism from a simple normal crossing pair (Y, Δ) to an algebraic variety X such that Δ is a boundary \mathbb{R} -divisor on Y . We assume that f is an isomorphism at the generic point of any stratum of the pair (Y, Δ) . Let L be a Cartier divisor on Y such that $L \sim_{\mathbb{R}, f} K_Y + \Delta$. Then $R^q f_* \mathcal{O}_Y(L) = 0$ for every $q > 0$.*

We note that Lemma 7.1 is an easy generalization of the relative Kawamata–Viehweg vanishing theorem. As an application of Lemma 7.1, we obtain Lemma 7.2. We have already used it several times in Section 5.

Lemma 7.2 (cf. [F13, Lemma 2.7]). *Let (V_1, D_1) and (V_2, D_2) be simple normal crossing pairs such that D_1 and D_2 are reduced. Let $f : V_1 \rightarrow V_2$*

be a proper morphism. Assume that there is a Zariski open subset U_1 (resp. U_2) of V_1 (resp. V_2) such that U_1 (resp. U_2) contains the generic point of any stratum of (V_1, D_1) (resp. (V_2, D_2)) and that f induces an isomorphism between U_1 and U_2 . Then $R^i f_* \omega_{V_1}(D_1) = 0$ for every $i > 0$ and $f_* \omega_{V_1}(D_1) \simeq \omega_{V_2}(D_2)$. By the Grothendieck duality, we obtain that $R^i f_* \mathcal{O}_{V_1}(-D_1) = 0$ for every $i > 0$ and $f_* \mathcal{O}_{V_1}(-D_1) \simeq \mathcal{O}_{V_2}(-D_2)$.

Proof. We can write

$$K_{V_1} + D_1 = f^*(K_{V_2} + D_2) + E$$

such that E is f -exceptional. We consider the following commutative diagram

$$\begin{array}{ccc} V_1^\nu & \xrightarrow{f^\nu} & V_2^\nu \\ \nu_1 \downarrow & & \downarrow \nu_2 \\ V_1 & \xrightarrow{f} & V_2 \end{array}$$

where $\nu_1 : V_1^\nu \rightarrow V_1$ and $\nu_2 : V_2^\nu \rightarrow V_2$ are the normalizations. We can write $K_{V_1^\nu} + \Theta_1 = \nu_1^*(K_{V_1} + D_1)$ and $K_{V_2^\nu} + \Theta_2 = \nu_2^*(K_{V_2} + D_2)$. By pulling back $K_{V_1} + D_1 = f^*(K_{V_2} + D_2) + E$ to V_1^ν by ν_1 , we have

$$K_{V_1^\nu} + \Theta_1 = (f^\nu)^*(K_{V_2^\nu} + \Theta_2) + \nu_1^* E.$$

Note that V_2^ν is smooth and Θ_2 is a reduced simple normal crossing divisor on V_2^ν . By the assumption, f^ν is an isomorphism over the generic point of any lc center of the pair (V_2^ν, Θ_2) (cf. 1.11). Therefore, $\nu_1^* E$ is effective since $K_{V_2^\nu} + \Theta_2$ is Cartier. Thus, we obtain that E is effective. We can easily check that f has connected fibers by the assumptions. Since V_2 satisfies Serre's S_2 condition, we can check that $\mathcal{O}_{V_2} \simeq f_* \mathcal{O}_{V_1}$ and $f_* \mathcal{O}_{V_1}(K_{V_1} + D_1) \simeq \mathcal{O}_{V_2}(K_{V_2} + D_2)$. On the other hand, we obtain $R^i f_* \mathcal{O}_{V_1}(K_{V_1} + D_1) = 0$ for every $i > 0$ by Lemma 7.1. Therefore, $Rf_* \mathcal{O}_{V_1}(K_{V_1} + D_1) \simeq \mathcal{O}_{V_2}(K_{V_2} + D_2)$ in the derived category of coherent sheaves on V_2 . Since V_1 and V_2 are Gorenstein, we have

$$\begin{aligned} Rf_* \mathcal{O}_{V_1}(-D_1) &\simeq R\mathcal{H}om(Rf_* \omega_{V_1}^\bullet(D_1), \omega_{V_2}^\bullet) \\ &\simeq R\mathcal{H}om(Rf_* \omega_{V_1}(D_1), \omega_{V_2}) \\ &\simeq R\mathcal{H}om(\omega_{V_2}(D_2), \omega_{V_2}) \simeq \mathcal{O}_{V_2}(-D_2) \end{aligned}$$

in the derived category of coherent sheaves on V_2 by the Grothendieck duality. Therefore, we have $R^i f_* \mathcal{O}_{V_1}(-D_1) = 0$ for every $i > 0$ and $f_* \mathcal{O}_{V_1}(-D_1) \simeq \mathcal{O}_{V_2}(-D_2)$. \square

Next, we prove the following theorem, which was proved for *embedded simple normal crossing pairs* in [F7, Theorem 2.39] and [F7, Theorem 2.47]. We note that we do not assume the existence of ambient spaces

in Theorem 7.3. However, we need the assumption that X is *quasi-projective*.

Theorem 7.3 (cf. [F7, Theorem 2.39 and Theorem 2.47]). *Let (X, B) be a quasi-projective simple normal crossing pair such that B is a boundary \mathbb{R} -divisor on X . Let $f : X \rightarrow Y$ be a proper morphism between algebraic varieties and let L be a Cartier divisor on X . Let q be an arbitrary integer. Then we have the following properties.*

- (i) *Assume that $L - (K_X + B)$ is f -semi-ample. Then every associated prime of $R^q f_* \mathcal{O}_X(L)$ is the generic point of the f -image of some stratum of (X, B) .*
- (ii) *Let $\pi : Y \rightarrow Z$ be a projective morphism. We assume that $L - (K_X + B) \sim_{\mathbb{R}} f^* A$ for some \mathbb{R} -Cartier \mathbb{R} -divisor A on Y such that A is nef and log big over Z with respect to $f : (X, B) \rightarrow Y$ (cf. [F7, Definition 2.46]). Then $R^q f_* \mathcal{O}_X(L)$ is π_* -acyclic, that is, $R^p \pi_* R^q f_* \mathcal{O}_X(L) = 0$ for every $p > 0$.*

Proof. Since X is quasi-projective, we can embed X into a smooth projective variety V . By Lemma 7.5 below, we can replace (X, B) and L with (X_k, B_k) and $\sigma^* L$ and assume that there exists an \mathbb{R} -divisor D on V such that $B = D|_X$. Then, by using Bertini's theorem, we can take a general complete intersection $W \subset V$ such that $\dim W = \dim X + 1$, $X \subset W$, and W is smooth at the generic point of every stratum of (X, B) (cf. the proof of [Ko5, Proposition 9.60]). We take a suitable resolution $\psi : M \rightarrow W$ with the following properties.

- (A) The strict transform X' of X is a simple normal crossing divisor on M .
- (B) We can write

$$K_{X'} + B' = \varphi^*(K_X + B) + E$$

such that $\varphi = \psi|_{X'}$, $(X', \text{Supp}(B' + E))$ is a globally embedded simple normal crossing pair (cf. Definition 2.13), B' is a boundary \mathbb{R} -divisor on X' , the φ -image of every stratum of (X', B') is a stratum of (X, B) , $\lceil E \rceil$ is effective and φ -exceptional.

- (C) φ is an isomorphism over the generic point of every stratum of (X, B) .
- (D) φ is an isomorphism at the generic point of every stratum of (X', B') .

Then

$$K_{X'} + B' + \{-E\} = \varphi^*(K_X + B) + \lceil E \rceil,$$

$$\varphi_* \mathcal{O}_{X'}(\varphi^* L + \lceil E \rceil) \simeq \mathcal{O}_X(L),$$

and

$$R^q \varphi_* \mathcal{O}_{X'}(\varphi^* L + {}^\Gamma E^\neg) = 0$$

for every $q > 0$ by Lemma 7.1. We note that

$$\varphi^* L + {}^\Gamma E^\neg - (K_{X'} + B' + \{-E\}) = \varphi^*(L - (K_X + B))$$

and that φ is an isomorphism at the generic point of every stratum of $(X', B' + \{-E\})$.

Therefore, by replacing (X, B) and L with $(X', B' + \{-E\})$ and $\varphi^* L + {}^\Gamma E^\neg$, we may assume that (X, B) is a quasi-projective globally embedded simple normal crossing pair (cf. Definition 2.13). In this case, the claims have already been established by [F7, Theorem 2.39] and [F7, Theorem 2.47]. \square

For some generalizations of Theorem 7.3 for *semi log canonical pairs*, see [F15].

Remark 7.4. Note that Theorem 7.3 (i) is contained in [F14, Theorem 1.1 (i)]. In [F14, Theorem 1.1], we do not assume that X is *quasi-projective*. On the other hand, we do not know how to remove the quasi-projectivity of X from Theorem 7.3 (ii).

By direct calculations, we can obtain the following elementary lemma. It was used in the proof of Theorem 7.3.

Lemma 7.5 (cf. [F7, Lemma 3.60]). *Let (X, B) be a simple normal crossing pair such that B is a boundary \mathbb{R} -divisor. Let V be a smooth variety such that $X \subset V$. Then we can construct a sequence of blow-ups*

$$V_k \rightarrow V_{k-1} \rightarrow \cdots \rightarrow V_0 = V$$

with the following properties.

- (1) $\sigma_{i+1} : V_{i+1} \rightarrow V_i$ is the blow-up along a smooth irreducible component of $\text{Supp} B_i$ for every $i \geq 0$.
- (2) We put $X_0 = X$, $B_0 = B$, and X_{i+1} is the strict transform of X_i for every $i \geq 0$.
- (3) We put $K_{X_{i+1}} + B_{i+1} = \sigma_{i+1}^*(K_{X_i} + B_i)$ for every $i \geq 0$.
- (4) There exists an \mathbb{R} -divisor D on V_k such that $D|_{X_k} = B_k$.
- (5) $\sigma_* \mathcal{O}_{X_k} \simeq \mathcal{O}_X$ and $R^q \sigma_* \mathcal{O}_{X_k} = 0$ for every $q > 0$, where $\sigma : V_k \rightarrow V_{k-1} \rightarrow \cdots \rightarrow V_0 = V$.

Proof. All we have to do is to check the property (5). We note that $\sigma_{i+1*} \mathcal{O}_{V_{i+1}}(K_{V_{i+1}}) \simeq \mathcal{O}_{V_{i+1}}(K_{V_{i+1}})$ and $R^q \sigma_{i+1*} \mathcal{O}_{V_{i+1}}(K_{V_{i+1}}) = 0$ for every q and for each step $\sigma_{i+1} : V_{i+1} \rightarrow V_i$ by Lemma 7.2. Therefore we obtain $R^q \sigma_* \mathcal{O}_{X_k}(K_{X_k}) = 0$ for every $q > 0$ and $\sigma_* \mathcal{O}_{X_k}(K_{X_k}) \simeq \mathcal{O}_X(K_X)$. Thus by the Grothendieck duality we obtain $R^q \sigma_* \mathcal{O}_{X_k} = 0$ for every $q > 0$ and $\sigma_* \mathcal{O}_{X_k} \simeq \mathcal{O}_X$ as in the proof of Lemma 7.2. \square

8. EXAMPLES

In this final section, we give supplementary examples for the Fujita–Kawamata semi-positivity theorem (cf. [Kw1, Theorem 5]), Viehweg’s weak positivity theorem, and the Fujino–Mori canonical bundle formula (cf. [FM]). For details of the original Fujita–Kawamata semi-positivity theorem, see, for example, [Mo, §5] and [F5, Section 5].

8.1 (Semi-positivity in the sense of Fujita–Kawamata). The following example is due to Takeshi Abe. It is a small remark on the definition of the Fujita–Kawamata semi-positivity: Definition 6.20.

Example 8.2. Let C be an elliptic curve and let E be a stable vector bundle on C such that the degree of E is -1 and the rank of E is two. Let $f_m : C \rightarrow C$ be the multiplication by m where m is a positive integer. In this case, every quotient line bundle L of E has non-negative degree. However, $\mathcal{O}_{\mathbb{P}(E)}(1)$ is not nef. It is because we can find a quotient line bundle M of f_m^*E whose degree is negative for some positive integer m .

8.3 (Canonical bundle formula). We give sample computations of our canonical bundle formula obtained in [FM]. We will freely use the notation in [FM]. For details of our canonical bundle formula, see [FM], [F2, §3], and [F3, §3, §4, §5, and §6].

8.4 (Kummer manifolds). Let E be an elliptic curve and let E^n be the n -times direct product of E . Let G be the cyclic group of order two of analytic automorphisms of E^n generated by an automorphism

$$g : E^n \rightarrow E^n : (z_1, \dots, z_n) \mapsto (-z_1, \dots, -z_n).$$

The automorphism g has 2^{2n} fixed points. Each singular point is terminal for $n \geq 3$ and is canonical for $n \geq 2$.

8.5 (Kummer surfaces). First, we consider $q : E^2/G \rightarrow E/G \simeq \mathbb{P}^1$, which is induced by the first projection, and $g = q \circ \mu : Y \rightarrow \mathbb{P}^1$, where $\mu : Y \rightarrow E^2/G$ is the minimal resolution of sixteen A_1 -singularities. It is easy to see that Y is a $K3$ surface. In this case, it is obvious that

$$g_*\mathcal{O}_Y(mK_{Y/\mathbb{P}^1}) \simeq \mathcal{O}_{\mathbb{P}^1}(2m)$$

for every $m \geq 1$. Thus, we can put $L_{Y/\mathbb{P}^1} = D$ for any degree two Weil divisor D on \mathbb{P}^1 . We obtain $K_Y = g^*(K_{\mathbb{P}^1} + L_{Y/\mathbb{P}^1})$. Let Q_i be the branch point of $E \rightarrow E/G \simeq \mathbb{P}^1$ for $1 \leq i \leq 4$. Then we have

$$L_{Y/\mathbb{P}^1}^{ss} = D - \sum_{i=1}^4 \left(1 - \frac{1}{2}\right) Q_i = D - \sum_{i=1}^4 \frac{1}{2} Q_i$$

by the definition of the semi-stable part L_{Y/\mathbb{P}^1}^{ss} . Therefore, we obtain

$$K_Y = g^* \left(K_{\mathbb{P}^1} + L_{Y/\mathbb{P}^1}^{ss} + \sum_{i=1}^4 \frac{1}{2} Q_i \right).$$

Thus,

$$L_{Y/\mathbb{P}^1}^{ss} = D - \sum_{i=1}^4 \frac{1}{2} Q_i \not\sim 0$$

but

$$2L_{Y/\mathbb{P}^1}^{ss} = 2D - \sum_{i=1}^4 Q_i \sim 0.$$

Note that L_{Y/\mathbb{P}^1}^{ss} is not a Weil divisor but a \mathbb{Q} -Weil divisor on \mathbb{P}^1 .

8.6 (Elliptic fibrations). Next, we consider E^3/G and E^2/G . We consider the morphism $p : E^3/G \rightarrow E^2/G$ induced by the projection $E^3 \rightarrow E^2 : (z_1, z_2, z_3) \rightarrow (z_1, z_2)$. Let $\nu : X' \rightarrow E^3/G$ be the weighted blow-up of E^3/G at sixty-four $\frac{1}{2}(1, 1, 1)$ -singularities. Thus

$$K_{X'} = \nu^* K_{E^3/G} + \sum_{j=1}^{64} \frac{1}{2} E_j,$$

where $E_j \simeq \mathbb{P}^2$ is the exceptional divisor for every j . Let P_i be an A_1 -singularity of E^2/G for $1 \leq i \leq 16$. Let $\psi : X \rightarrow X'$ be the blow-up of X' along the strict transform of $p^{-1}(P_i)$, which is isomorphic to \mathbb{P}^1 , for every i . Then we obtain the following commutative diagram.

$$\begin{array}{ccc} E^3/G & \xleftarrow{\phi := \nu \circ \psi} & X \\ p \downarrow & & \downarrow f \\ E^2/G & \xleftarrow[\mu]{} & Y \end{array}$$

Note that

$$K_X = \phi^* K_{E^3/G} + \sum_{j=1}^{64} \frac{1}{2} E_j + \sum_{k=1}^{16} F_k,$$

where E_j is the strict transform of E_j on X and F_k is the ψ -exceptional prime divisor for every k . We can check that X is a smooth projective threefold. We put $C_i = \mu^{-1}(P_i)$ for every i . It can be checked that C_i is a (-2) -curve for every i . It is easily checked that f is smooth outside $\sum_{i=1}^{16} C_i$ and that the degeneration of f is of type I_0^* along C_i for every i . We renumber $\{E_j\}_{j=1}^{64}$ as $\{E_i^j\}$, where $f(E_i^j) = C_i$ for

every $1 \leq i \leq 16$ and $1 \leq j \leq 4$. We note that f is flat since f is equi-dimensional.

Let us recall the following theorem (cf. [Kw2, Theorem 20] and [N2, Corollary 3.2.1 and Theorem 3.2.3]).

Theorem 8.7 (... , Kawamata, Nakayama, ...). *We have the following isomorphism.*

$$(f_*\omega_{X/Y})^{\otimes 12} \simeq \mathcal{O}_Y \left(\sum_{i=1}^{16} 6C_i \right),$$

where $\omega_{X/Y} \simeq \mathcal{O}_X(K_{X/Y}) = \mathcal{O}_X(K_X - f^*K_Y)$.

The proof of Theorem 8.7 depends on the investigation of the upper canonical extension of the Hodge filtration and the period map. It is obvious that

$$2K_X = f^* \left(2K_Y + \sum_{i=1}^{16} C_i \right)$$

and

$$2mK_X = f^* \left(2mK_Y + m \sum_{i=1}^{16} C_i \right)$$

for all $m \geq 1$ since $f^*C_i = 2F_i + \sum_{j=1}^4 E_i^j$ for every i . Therefore, we have $2L_{X/Y} \sim \sum_{i=1}^{16} C_i$. On the other hand, $f_*\omega_{X/Y} \simeq \mathcal{O}_Y(\lfloor L_{X/Y} \rfloor)$. Note that Y is a smooth surface and f is flat. Since

$$\mathcal{O}_Y(12\lfloor L_{X/Y} \rfloor) \simeq (f_*\omega_{X/Y})^{\otimes 12} \simeq \mathcal{O}_Y \left(\sum_{i=1}^{16} 6C_i \right),$$

we have

$$12L_{X/Y} \sim 6 \sum_{i=1}^{16} C_i \sim 12\lfloor L_{X/Y} \rfloor.$$

Thus, $L_{X/Y}$ is a Weil divisor on Y . It is because the fractional part $\{L_{X/Y}\}$ is effective and linearly equivalent to zero. So, $L_{X/Y}$ is numerically equivalent to $\frac{1}{2} \sum_{i=1}^{16} C_i$. We have $g^*Q_i = 2G_i + \sum_{j=1}^4 C_i^j$ for every i . Here, we renumbered $\{C_j\}_{j=1}^{16}$ as $\{C_i^j\}_{i,j=1}^4$ such that $g(C_i^j) = Q_i$ for every i and j . More precisely, we put $2G_i = g^*Q_i - \sum_{j=1}^4 C_i^j$ for every i . We note that we used notations in 8.5. We consider $A := g^*D - \sum_{i=1}^4 G_i$. Then A is a Weil divisor and $2A \sim \sum_{i=1}^{16} C_i$. Thus, A is numerically equivalent to $\frac{1}{2} \sum_{i=1}^{16} C_i$. Since $H^1(Y, \mathcal{O}_Y) = 0$,

we can put $L_{X/Y} = A$. So, we have

$$L_{X/Y}^{ss} = g^*D - \sum_{i=1}^4 G_i - \sum_{j=1}^{16} \frac{1}{2} C_j.$$

We obtain the following canonical bundle formula.

Theorem 8.8. *The next formula holds.*

$$K_X = f^* \left(K_Y + L_{X/Y}^{ss} + \sum_{j=1}^{16} \frac{1}{2} C_j \right),$$

where $L_{X/Y}^{ss} = g^*D - \sum_{i=1}^4 G_i - \sum_{j=1}^{16} \frac{1}{2} C_j$.

We note that $2L_{X/Y}^{ss} \sim 0$ but $L_{X/Y}^{ss} \not\sim 0$. The semi-stable part $L_{X/Y}^{ss}$ is not a Weil divisor but a \mathbb{Q} -divisor on Y .

The next lemma is obvious since the index of $K_{E^3/G}$ is two. We give a direct proof here.

Lemma 8.9. $H^0(Y, L_{X/Y}) = 0$.

Proof. If there exists an effective Weil divisor B on Y such that $L_{X/Y} \sim B$. Since $B \cdot C_i = -1$, we have $B \geq \frac{1}{2} C_i$ for all i . Thus $B \geq \sum_{i=1}^{16} \frac{1}{2} C_i$. This implies that $B - \sum_{i=1}^{16} \frac{1}{2} C_i$ is an effective \mathbb{Q} -divisor and is numerically equivalent to zero. Thus $B = \sum_{i=1}^{16} \frac{1}{2} C_i$. It is a contradiction. \square

We can easily check the following corollary.

Corollary 8.10. *We have*

$$f_* \omega_{X/Y}^{\otimes m} \simeq \begin{cases} \mathcal{O}_Y(\sum_{i=1}^{16} n C_i) & \text{if } m = 2n, \\ \mathcal{O}_Y(L_{X/Y} + \sum_{i=1}^{16} n C_i) & \text{if } m = 2n + 1. \end{cases}$$

In particular, $f_* \omega_{X/Y}^{\otimes m}$ is not nef for any $m \geq 1$. We can also check that

$$H^0(Y, f_* \omega_{X/Y}^{\otimes m}) \simeq \begin{cases} \mathbb{C} & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

Corollary 8.10 shows that [T, Theorem 1.9 (1)] is sharp.

8.11 (Weak positivity). Let us recall the definition of Viehweg's weak positivity (cf. [V1, Definition 1.2] and [V3, Definition 2.11]). The reader can find some interesting applications of a generalization of Viehweg's weak positivity theorem in [FG].

Definition 8.12 (Weak positivity). Let W be a smooth quasi-projective variety and let \mathcal{F} be a locally free sheaf on W . Let U be an open subvariety of W . Then, \mathcal{F} is *weakly positive over U* if for every ample invertible sheaf \mathcal{H} and every positive integer α there exists some positive integer β such that $S^{\alpha\cdot\beta}(\mathcal{F}) \otimes \mathcal{H}^\beta$ is generated by global sections over U . This means that the natural map

$$H^0(W, S^{\alpha\cdot\beta}(\mathcal{F}) \otimes \mathcal{H}^\beta) \otimes \mathcal{O}_W \rightarrow S^{\alpha\cdot\beta}(\mathcal{F}) \otimes \mathcal{H}^\beta$$

is surjective over U .

Remark 8.13 (cf. [V1, (1.3) Remark. iii]). In Definition 8.12, it is enough to check the condition for one invertible sheaf \mathcal{H} , not necessarily ample, and all $\alpha > 0$. For details, see [V3, Lemma 2.14 a)].

Remark 8.14. In [V2, Definition 3.1], $S^{\alpha\cdot\beta}(\mathcal{F}) \otimes \mathcal{H}^{\otimes\beta}$ is only required to be generically generated. See also [Mo, (5.1) Definition].

We explicitly check the weak positivity for the elliptic fibration constructed in 8.6 (cf. [V1, Theorem 4.1 and Theorem III] and [V3, Theorem 2.41 and Corollary 2.45]).

Proposition 8.15. *Let m be a positive integer. Let $f : X \rightarrow Y$ be the elliptic fibration constructed in 8.6. Then $f_*\omega_{X/Y}^{\otimes m}$ is weakly positive over $Y_0 = Y \setminus \sum_{i=1}^{16} C_i$. Let U be a Zariski open set such that $U \not\subset Y_0$. Then $f_*\omega_{X/Y}^{\otimes m}$ is not weakly positive over U .*

Proof. Let H be a very ample Cartier divisor on Y such that $L_{X/Y} + H$ is very ample. We put $\mathcal{H} = \mathcal{O}_Y(H)$. Let α be an arbitrary positive integer. Then

$$S^\alpha(f_*\omega_{X/Y}^{\otimes m}) \otimes \mathcal{H} \simeq \mathcal{O}_Y \left(\alpha \sum_{i=1}^{16} nC_i + H \right)$$

if $m = 2n$. When $m = 2n + 1$, we have

$$\begin{aligned} & S^\alpha(f_*\omega_{X/Y}^{\otimes m}) \otimes \mathcal{H} \\ & \simeq \begin{cases} \mathcal{O}_Y(\alpha \sum_{i=1}^{16} nC_i + H + L_{X/Y} + \lfloor \frac{\alpha}{2} \rfloor \sum_{i=1}^{16} C_i) & \text{if } \alpha \text{ is odd,} \\ \mathcal{O}_Y(\alpha \sum_{i=1}^{16} nC_i + H + \frac{\alpha}{2} \sum_{i=1}^{16} C_i) & \text{if } \alpha \text{ is even.} \end{cases} \end{aligned}$$

Thus, $S^\alpha(f_*\omega_{X/Y}^{\otimes m}) \otimes \mathcal{H}$ is generated by global sections over Y_0 for every $\alpha > 0$. Therefore, $f_*\omega_{X/Y}^{\otimes m}$ is weakly positive over Y_0 .

Let \mathcal{A} be an ample invertible sheaf on Y . We put $k = \max_j (C_j \cdot \mathcal{A})$. Let α be a positive integer with $\alpha > k/2$. We note that

$$S^{2\alpha\cdot\beta}(f_*\omega_{X/Y}^{\otimes m}) \otimes \mathcal{A}^{\otimes\beta} \simeq \left(\mathcal{O}_Y(\alpha \sum_{i=1}^{16} mC_i) \otimes \mathcal{A} \right)^{\otimes\beta}.$$

If $H^0(Y, S^{2\alpha\cdot\beta}(f_*\omega_{X/Y}^{\otimes m}) \otimes \mathcal{A}^{\otimes\beta}) \neq 0$, then we can take

$$G \in \left| \left(\mathcal{O}_Y(\alpha \sum_{i=1}^{16} mC_i) \otimes \mathcal{A} \right)^{\otimes\beta} \right|.$$

In this case, $G \cdot C_i < 0$ for every i because $\alpha > k/2$. Therefore, we obtain $G \geq \sum_{i=1}^{16} C_i$. Thus, $S^{2\alpha\cdot\beta}(f_*\omega_{X/Y}^{\otimes m}) \otimes \mathcal{A}^{\otimes\beta}$ is not generated by global sections over U for any $\beta \geq 1$. This means that $f_*\omega_{X/Y}^{\otimes m}$ is not weakly positive over U . \square

Proposition 8.15 implies that [V3, Corollary 2.45] is the best result.

Example 8.16. Let $f : X \rightarrow Y$ be the elliptic fibration constructed in 8.6. Let $Z := C \times X$, where C is a smooth projective curve with the genus $g(C) = r \geq 2$. Let $\pi_1 : Z \rightarrow C$ (resp. $\pi_2 : Z \rightarrow X$) be the first (resp. second) projection. We put $h := f \circ \pi_2 : Z \rightarrow Y$. In this case, $K_Z = \pi_1^*K_C \otimes \pi_2^*K_X$. Therefore, we obtain

$$h_*\omega_{Z/Y}^{\otimes m} = f_*\pi_{2*}(\pi_1^*\omega_C^{\otimes m} \otimes \pi_2^*\omega_X^{\otimes m}) \otimes \omega_Y^{\otimes -m} = (f_*\omega_{X/Y}^{\otimes m})^{\oplus l},$$

where $l = \dim H^0(C, \mathcal{O}_C(mK_C))$. Thus, $l = (2m - 1)r - 2m + 1$ if $m \geq 2$ and $l = r$ if $m = 1$. So, $h_*\omega_{Z/Y}$ is a rank $r \geq 2$ vector bundle on Y such that $h_*\omega_{Z/Y}$ is not semi-positive. We note that h is smooth over $Y_0 = Y \setminus \sum_{i=1}^{16} C_i$. We also note that $h_*\omega_{Z/Y}^{\otimes m}$ is weakly positive over Y_0 for every $m \geq 1$ by [V3, Theorem 2.41 and Corollary 2.45].

Example 8.16 shows that the assumption on the local monodromies around $\sum_{i=1}^{16} C_i$ is indispensable for the Fujita–Kawamata semi-positivity theorem (cf. [Kw1, Theorem 5 (iii)]).

We close this section with a comment on [FM].

8.17 (Comment). We give a remark on [FM, Section 4]. In [FM, 4.4], $g : Y \rightarrow X$ is a log resolution of (X, Δ) . However, it is better to assume that g is a log resolution of $(X, \Delta - (1/b)B^\Delta)$ for the proof of [FM, Theorem 4.8].

REFERENCES

- [Be] B. Berndtsson, Curvature of vector bundles associated to holomorphic fibrations, *Ann. of Math. (2)* **169** (2009), no. 2, 531–560.
- [BeP] B. Berndtsson, M. Păun, Bergman kernels and the pseudoeffectivity of relative canonical bundles, *Duke Math. J.* **145** (2008), no. 2, 341–378.
- [BiM] E. Bierstone, P. D. Milman, Resolution except for minimal singularities I, preprint (2011).
- [BiP] E. Bierstone, F. Vera Pacheco, Resolution of singularities of pairs preserving semi-simple normal crossings, preprint (2011).
- [Bo] A. Borel et al., *Algebraic D-modules*, Perspectives in Mathematics, **2**, Academic Press, Inc., Boston, MA, 1987.
- [BZ] J.-L. Brylinski, S. Zucker, An overview of recent advances in Hodge theory, *Several complex variables*, VI, 39–142, *Encyclopaedia Math. Sci.*, **69**, Springer, Berlin, 1990.
- [CK] E. Cattani, A. Kaplan, Polarized mixed Hodge structures and the local monodromy of a variation of Hodge structure, *Invent. Math.* **67** (1982), no. 1, 101–115.
- [CKS] E. Cattani, A. Kaplan, W. Schmid, Degeneration of Hodge structures, *Ann. of Math. (2)* **123** (1986), no. 3, 457–535.
- [D1] P. Deligne, *Equations Différentielles à Points Singuliers Réguliers*, Lecture Notes in Math., **163**, Springer-Verlag, 1970.
- [D2] P. Deligne, Théorie de Hodge II, *Inst. Hautes Études Sci. Publ. Math.* **40** (1971), 5–57.
- [D3] P. Deligne, Théorie de Hodge III, *Inst. Hautes Études Sci. Publ. Math.* **44** (1972), 5–77.
- [dB] P. Du Bois, Structure de Hodge mixte sur la cohomologie évanescence, *Ann. Inst. Fourier (Grenoble)* **35** (1985), no. 1, 191–213.
- [E1] F. El Zein, Théorie de Hodge des cycles évanescents, *Ann. Sci. École Norm. Sup. (4)* **19** (1986), no. 1, 107–184.
- [E2] F. El Zein, *Introduction à la théorie de Hodge mixte*, *Actualités Mathématiques*, Hermann, Paris, 1991.
- [FGAE] B. Fantechi, L. Göttsch, L. Illusie, S.L. Kleinman, N. Nitsure, A. Vistoli, *Fundamental Algebraic Geometry*, *Mathematical Surveys and Monographs*, **123**, American Math. Soc., 2005.
- [F1] O. Fujino, Abundance theorem for semi log canonical threefolds, *Duke Math. J.* **102** (2000), no. 3, 513–532.
- [F2] O. Fujino, Algebraic fiber spaces whose general fibers are of maximal Albanese dimension, *Nagoya Math. J.* **172** (2003), 111–127.
- [F3] O. Fujino, A canonical bundle formula for certain algebraic fiber spaces and its applications, *Nagoya Math. J.* **172** (2003), 129–171.
- [F4] O. Fujino, Higher direct images of log canonical divisors, *J. Differential Geom.* **66** (2004), no. 3, 453–479.
- [F5] O. Fujino, Remarks on algebraic fiber spaces, *J. Math. Kyoto Univ.* **45** (2005), no. 4, 683–699.
- [F6] O. Fujino, What is log terminal?, in *Flips for 3-folds and 4-folds* (Alessio Corti, ed.), 49–62, Oxford University Press, 2007.
- [F7] O. Fujino, Introduction to the log minimal model program for log canonical pairs, preprint (2009).

- [F8] O. Fujino, On injectivity, vanishing and torsion-free theorems for algebraic varieties, *Proc. Japan Acad. Ser. A Math. Sci.* **85** (2009), no. 8, 95–100.
- [F9] O. Fujino, Theory of non-lc ideal sheaves: basic properties, *Kyoto Journal of Mathematics*, Vol. **50**, No. 2 (2010), 225–245.
- [F10] O. Fujino, Introduction to the theory of quasi-log varieties, *Classification of algebraic varieties*, 289–303, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011,
- [F11] O. Fujino, Fundamental theorems for the log minimal model program, *Publ. Res. Inst. Math. Sci.* **47** (2011), no. 3, 727–789.
- [F12] O. Fujino, Minimal model theory for log surfaces, to appear in *Publ. Res. Inst. Math. Sci.*
- [F13] O. Fujino, On isolated log canonical singularities with index one, *J. Math. Sci. Univ. Tokyo* **18** (2011), 299–323.
- [F14] O. Fujino, Vanishing theorems, preprint (2012).
- [F15] O. Fujino, Fundamental theorems for semi log canonical pairs, preprint (2012).
- [FG] O. Fujino, Y. Gongyo, On images of weak Fano manifolds II, preprint (2012).
- [FM] O. Fujino, S. Mori, A canonical bundle formula, *J. Differential Geom.* **56** (2000), no. 1, 167–188.
- [Ft] T. Fujita, On Kähler fiber spaces over curves, *J. Math. Soc. Japan* **30** (1978), no. 4, 779–794.
- [Fu] W. Fulton, *Intersection theory*, Second edition. *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, **2**. Springer-Verlag, Berlin, 1998.
- [Gf] P. A. Griffiths, Periods of integrals on algebraic manifolds. III. Some global differential-geometric properties of the period mapping, *Inst. Hautes Études Sci. Publ. Math. No.* **38** 1970 125–180.
- [Go] A. Grothendieck, *Éléments de géométrie algébrique. IV, Étude locale des schémas et des morphismes de schémas IV.* (French) *Inst. Hautes Études Sci. Publ. Math. No.* **32** 1967.
- [GN] F. Guillén, V. Navarro Aznar, Sur le théorème local des cycles invariants, *Duke Math. J.* **61** (1990), no. 1, 133–155.
- [GNPP] F. Guillén, V. Navarro Aznar, P. Pascual Gainza, F. Puerta, *Hyperrésolutions cubiques et descente cohomologique*, *Lecture Notes in Mathematics*, **1335**. Springer-Verlag, Berlin, 1988.
- [Ks] M. Kashiwara, A study of variation of mixed Hodge structure, *Publ. Res. Inst. Math. Sci.* **22** (1986), no. 5, 991–1024.
- [Kt] N. Katz, An Overview of Deligne’s Work on Hilbert’s Twenty-First Problem, *Proceedings of Symposia in Pure Mathematics* **28**, (1976), 537–557
- [KO] N. Katz, T. Oda, On the differentiation of De Rham cohomology classes with respect to parameters, *J. Math. Kyoto Univ.* **8** (1968), 199–213.
- [Kw1] Y. Kawamata, Characterization of abelian varieties, *Compositio Math.* **43** (1981), no. 2, 253–276.
- [Kw2] Y. Kawamata, Kodaira dimension of certain algebraic fiber spaces, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **30** (1983), no. 1, 1–24.

- [Kw3] Y. Kawamata, Semipositivity theorem for reducible algebraic fiber spaces, preprint (2009).
- [Ko1] J. Kollár, Higher direct images of dualizing sheaves, I. *Ann. of Math.* **123** (1986), 11–42.
- [Ko2] J. Kollár, Higher direct images of dualizing sheaves, II. *Ann. of Math.* **124** (1986), 171–202.
- [Ko3] J. Kollár, Subadditivity of the Kodaira dimension: fibers of general type, *Algebraic geometry*, Sendai, 1985, 361–398, *Adv. Stud. Pure Math.*, **10**, North-Holland, Amsterdam, 1987.
- [Ko4] J. Kollár, Kodaira’s canonical bundle formula and adjunction, *Flips for 3-folds and 4-folds*, 134–162, *Oxford Lecture Ser. Math. Appl.*, **35**, Oxford Univ. Press, Oxford, 2007.
- [Ko5] J. Kollár, *Singularities of the Minimal Model Program*, preprint (2011).
- [MT] C. Mourougane, S. Takayama, Extension of twisted Hodge metrics for Kähler morphisms, *J. Differential Geom.* **83** (2009), no. 1, 131–161.
- [Mo] S. Mori, Classification of higher-dimensional varieties, *Algebraic geometry*, Bowdoin, 1985 (Brunswick, Maine, 1985), 269–331, *Proc. Sympos. Pure Math.*, **46**, Part 1, Amer. Math. Soc., Providence, RI, 1987.
- [Mu1] D. Mumford, *Lectures on curves on an algebraic surface*, With a section by G. M. Bergman. *Annals of Mathematics Studies*, No. **59** Princeton University Press, Princeton, N.J. 1966.
- [Mu2] D. Mumford, *Abelian Varieties*, Oxford University Press, 1970.
- [N1] N. Nakayama, Hodge filtrations and the higher direct images of canonical sheaves, *Invent. Math.* **85** (1986), no. 1, 217–221.
- [N2] N. Nakayama, Local structure of an elliptic fibration, *Higher dimensional birational geometry* (Kyoto, 1997), 185–295, *Adv. Stud. Pure Math.*, **35**, Math. Soc. Japan, Tokyo, 2002.
- [NA] V. Navarro Aznar, Sur la théorie de Hodge–Deligne, *Invent. Math.* **90** (1987), no. 1, 11–76.
- [P] G. J. Pearlstein, private note, 9 July, 2010.
- [PS] C. Peters, J. Steenbrink, *Mixed Hodge structures*, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, **52**. Springer-Verlag, Berlin, 2008.
- [Sa1] M. Saito, Modules de Hodge polarisables, *Publ. Res. Inst. Math. Sci.* **24** (1988), no. 6, 849–995.
- [Sa2] M. Saito, Mixed Hodge Modules, *Publ. Res. Inst. Math. Sci.* **26** (1990), no. 2, 221–333.
- [Sc] W. Schmid, Variation of Hodge structure: the singularities of the period mapping, *Invent. Math.* **22** (1973), 211–319.
- [SSU] M-H, Saito, Y. Shimizu, S. Usui, Variation of mixed Hodge structure and the Torelli problem, *Algebraic geometry*, Sendai, 1985, 649–693, *Adv. Stud. Pure Math.*, **10**, North-Holland, Amsterdam, 1987.
- [St1] J. Steenbrink, Limits of Hodge structures, *Invent. Math.* **31** (1975/76), no. 3, 229–257.
- [St2] J. Steenbrink, Mixed Hodge structure on the vanishing cohomology, *Real and complex singularities* (Proc. Ninth Nordic Summer School/NAVF

- Sympos. Math., Oslo, 1976), pp. 525–563. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.
- [SZ] J. Steenbrink, S. Zucker, Variation of mixed Hodge structure, I. Invent. Math. **80** (1985), no. 3, 489–542.
 - [T] H. Tsuji, Global generation of the direct images of relative pluricanonical systems, preprint (2010).
 - [U] S. Usui, Mixed Torelli problem for Todorov surfaces, Osaka J. Math. **28** (1991) 697–735
 - [V1] E. Viehweg, Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces, Algebraic varieties and analytic varieties (Tokyo, 1981), 329–353, Adv. Stud. Pure Math., **1**, North-Holland, Amsterdam, 1983.
 - [V2] E. Viehweg, Weak positivity and the additivity of the Kodaira dimension. II. The local Torelli map, Classification of algebraic and analytic manifolds (Katata, 1982), 567–589, Progr. Math., **39**, Birkhäuser Boston, Boston, MA, 1983.
 - [V3] E. Viehweg, *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], **30**. Springer-Verlag, Berlin, 1995.
 - [Z] S. Zucker, Remarks on a theorem of Fujita, J. Math. Soc. Japan, **34** (1982), 47–54.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KYOTO UNIVERSITY,
KYOTO 606-8502, JAPAN

E-mail address: fujino@math.kyoto-u.ac.jp

TOKYO DENKI UNIVERSITY, SCHOOL OF ENGINEERING, DEPARTMENT OF
MATHEMATICS, TOKYO, JAPAN

E-mail address: fujisawa@mail.dendai.ac.jp